On Algebraic Structures on $\kappa$-Intuitionistic $Q$–Fuzzy Quotient Group

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Abstract

In this paper, we introduced of $\kappa$–Intuitionistic $Q$–fuzzy quotient group, $\kappa$–Intuitionistic $Q$–fuzzy cosets of an Intuitionistic $Q$–fuzzy normal subgroup are defined and discussed. A homomorphism from a given group onto the set of all $\kappa$–intuitionistic $Q$–fuzzy quotient group is established. Some related results has been derived.

Keywords: Intuitionistic Fuzzy Set (IFS); Intuitionistic Fuzzy Subset (IFSb); Intuitionistic Fuzzy Subgroup (IFSG); Intuitionistic Fuzzy Normal Subgroup (IFNSG); $\kappa$–Intuitionistic $Q$–fuzzy Set ($\kappa$–IQFS); $\kappa$–Intuitionistic $Q$–fuzzy subset ($\kappa$–IQFBS); $\kappa$–Intuitionistic $Q$–fuzzy subgroup ($\kappa$–IQFSG); $\kappa$–Intuitionistic $Q$–fuzzy normal subgroup ($\kappa$–IQFNSG); $\kappa$–Intuitionistic $Q$–fuzzy coset; $\kappa$–Intuitionistic $Q$–fuzzy quotient group.

1. Introduction

The fundamental concept of fuzzy sets was initiated by Zadeh L[21] in 1965. Since then these ideas have been applied to other algebraic structures such as groups, rings, modules, vector spaces and topologies. In 1986, introduced the new notation on Intuitionistic Fuzzy Sets by Atanassov K T[1], Barbhuiya S R[4],
introduced the concept of t- Intuitionistic Fuzzy Sub algebra of BG-Algebras in 2015.


In this paper, we introduce the notion of κ-Intuitionistic Q−Fuzzy cosets of an intuitionistic Q−fuzzy normal subgroup and κ-Intuitionistic Q−Fuzzy Quotient Group and discuss some of their properties.

2. Preliminaries

In this section, we cite the fundamental definitions that will be used in the sequel.

2.1 Definition [ZadehL A (20)]

Let X be a non-empty set. A FSB of the set X is a mapping μ: X → [0, 1].

2.2. Definition [Atanassov K T(1)]

Let X be a fixed non-empty set. An IFS A of X is an object of the following form from A = {< x, μA(x), νA(x) >: x ∈ X}, where μA : X → [0, 1] and νA : X → [0, 1] define the degree of membership and degree of non-
membership of the element \(x \in X\) respectively and for any \(x \in X\), we have \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\).

2.2.1 Remark

(i) When \(\mu_A(x) + \nu_A(x) = 1\), i.e., when \(\nu_A(x) = 1 - \mu_A(x) = \mu_A^c(x)\). Then \(A\) is called \(FS\).

(ii) We use the notation \(A = (\mu_A, \nu_A)\) to denote the \(IFS\) \(A\) of \(X\).

2.3 Definition [Sharma P K (11)]

Let \(G\) be a group. An \(IFS\) \(A = (\mu_A, \nu_A)\) of \(G\) is called \(IFSG\) of \(G\) if

(i) \(\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)\)

(ii) \(\mu_A(x^{-1}) = \mu_A(x)\)

(iii) \(\nu_A(xy) \leq \nu_A(x) \lor \nu_A(y)\)

(iv) \(\nu_A(x^{-1}) = \nu_A(x), \forall x, y \in G\)

or Equivalently \(A\) is an \(IFSG\) of \(G\) if and only if \(\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)\) and \(\nu_A(xy) \leq \nu_A(x) \lor \nu_A(y)\).

2.4 Definition [Sharma P K (11)]

An \(IFSG\) \(A = (\mu_A, \nu_A)\) of a group \(G\) is said to be \(IFNSG\) of \(G\) if

(i) \(\mu_A(xy) = \mu_A(yx)\)

(ii) \(\nu_A(xy) = \nu_A(yx), \forall x, y \in G\)

Or Equivalently \(A\) is an \(IFNSG\) of a group \(G\) is normal if and only if \(\mu_A(y^{-1}xy) \geq \mu_A(x)\) and \(\nu_A(y^{-1}xy) \leq \nu_A(x), \forall x, y \in G\).

2.5 Definition [Sharma P K (10)]

Let \(A\) be \(IFS\) of a universe set \(X\). Then \((\alpha, \beta)\)-cut of \(A\) is a crisp subset \(C_{\alpha, \beta}(A)\) of the \(IFS\) \(A\) is given by \(C_{\alpha, \beta}(A) = \{x \in X\ such\ that\ \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\ \}, \ where\ \alpha, \beta \in [0,1] \) with \(\alpha + \beta \leq 1\).

2.6 Theorem

If \(A\) is \(IFS\) of a group \(G\). Then \(A\) is \(IFSG(IFNSG)\) of \(G\) if and only if \(C_{\alpha, \beta}(A)\) is a subgroup (normal) of group \(G\), for all \(\alpha, \beta \in [0,1]\) with \(\alpha + \beta \leq 1\).

2.7 Definition[Solairaju A and Nagarajan R (16)]

Let \(Q\) and \(G\) a set and a group respectively. A mapping \(\mu: G \times Q \to [0,1]\) is named \(Q - FS\) in \(G\). For any \(Q - FS\mu\) in \(G\) and \(t \in [0,1]\) we define the set \(U(\mu; t) = \{x \in G/ \mu(x, q) \geq t, q \in Q\}\) which is named an upper cut of “\(\mu\)” and may be use to the characterization of \(\mu\).
2.8 Definition [Solairaju A and Nagarajan R (16)]

A $Q - FS\mu$ is named $Q - FSG$ of $G$ if

1. $\mu(xy, q) \geq \min\{\mu(x, q), \mu(y, q)\}$
2. $\mu(x^{-1}, q) = \mu(x, q)$
3. $\mu(x, q) = 1$, for all $x, y \in G$ and $q \in Q$.

3. On algebraic structures on $\kappa$-Intuitionistic $Q - Fuzzy$ Quotient Group

3.1 Definition

Let $A$ be an IQFNSG of a group $H$. Let $\kappa \in [0,1]$ and $q \in Q$. For any $m \in H$ define an IQFSA$^\kappa_m$ of $H$ is called $\kappa$-Intuitionistic $q - fuzzy$ Coset of $A$ in $H$ as follows:

$A^\kappa_m(g, q) = \{\theta^\kappa_{A^\kappa_m}(g, q), \emptyset^\kappa_{A^\kappa_m}(g, q)\}$,

where

$\theta^\kappa_{A^\kappa_m}(g, q) = \{\theta_A(gm^{-1}, q)\land \kappa\}$ and $\emptyset^\kappa_{A^\kappa_m}(g, q) = \{\emptyset_A(gm^{-1}, q)\land 1 - \kappa\}$,

$\forall m, g \in H$ and $q \in Q$.

3.2 Proposition

Let $S$ and $Q$ be the set of all $\kappa$-Intuitionistic $q - fuzzy$ cosets of an IQFNSGA in $H$. i.e., $S = \{A^\kappa_m: m \in H$ and $q \in Q\}$. Then the binary operations $\otimes$ defined on the set $S$ as follows:

$A^\kappa_m \otimes A^\kappa_n = A^\kappa_{mn}, \forall m, n \in H$ and $q \in Q$ is a well defined operation.

Proof:

Let $A^\kappa_m = A^\kappa_{m'}, A^\kappa_n = A^\kappa_{n'},$ for some $m, n, m', n' \in H$ and $q \in Q$.

Let $g \in H$ and $q \in Q$ be any element, then

$[A^\kappa_m \otimes A^\kappa_n](g, q) = (A^\kappa_{mn})(g, q) = (\theta^\kappa_{A^\kappa_{mn}}(g, q), \emptyset^\kappa_{A^\kappa_{mn}}(g, q))$.

Now

$\theta^\kappa_{A^\kappa_{mn}}(g, q) = \{\theta_A((mn)^{-1}, q)\land \kappa\}$

$= \theta^\kappa_{A^\kappa_m}(gm^{-1}, q) = \theta^\kappa_{A^\kappa_n}(gm^{-1}, q)$

$= \theta_A((gn^{-1})^{-1}, q) = \theta_A((gm^{-1})^{-1}, q)\land \kappa$.

Similarly, we can show that

$\emptyset^\kappa_{A^\kappa_{mn}}(g, q) = \emptyset^\kappa_{A^\kappa_{m'}}(g, q), \forall g \in H$ and $q \in Q$.

Therefore $\otimes$ is well defined operation on $S$. 

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3.3 Lemma

If $A$ is IQFNSG of a group $G$. Then $A^k_m = A^k_{m'}$ $\iff$ $Nm = Nm'$, $\forall m, m' \in H$ and $q \in Q$. where $N = C_{\kappa,1-\kappa}(A, q)$.

Proof:

Let $A^k_m = A^k_{m'}$ for $m, m' \in H$ and $q \in Q$

then $A^k_m(m', q) = A^k_{m'}(m', q)$

$\left\{ \theta_A(m', q) \right\} \cap \kappa, \{ \varnothing_A(m', q) \cap 1 - \kappa \}$

$= \left\{ \theta_A(m', q) \right\} \cap \kappa, \{ \varnothing_A(m', q) \cap 1 - \kappa \}$

and $m' \in C_{\kappa,1-\kappa}(A) = N$

$\Rightarrow Nm = Nm' \tag{1}$

Now, we show that if $Nm = Nm'$, then $A^k_m = A^k_{m'}$.

Let for some $n \in H$ and $q \in Q, A^k_m(n, q) \neq A^k_{m'}(n, q)$

i.e., $\left\{ \theta_A(mn^{-1}, q) \cap \kappa \right\}, \{ \varnothing_A(mn^{-1}, q) \cap 1 - \kappa \}$

Suppose $\theta_A(mn^{-1}, q) < \kappa$ and $\theta_A(m', n^{-1}, q) \geq \kappa$

Therefore $\varnothing_A(mn^{-1}, q) \leq 1 - \kappa$ and $\varnothing_A(mn^{-1}, q) \leq 1 - \kappa$

$\Rightarrow m n^{-1} \in C_{\kappa,1-\kappa}(A) = N$

$\Rightarrow Nm n^{-1} = N \Rightarrow Nmn^{-1} = N$ (Using 1) $\Rightarrow mn^{-1} \in N$

and so $\kappa(\theta_A(mn^{-1}, q) \geq \kappa$, a contradiction.

Similarly, if $\theta_A(mn^{-1}, q) \geq \kappa$ and $\theta_A(m', n^{-1}, q) < \kappa$ also leads to contradiction. Therefore either $\theta_A(mn^{-1}, q) \geq \kappa$ and $\theta_A(m', n^{-1}, q) \geq \kappa$ i.e., $\varnothing_A(mn^{-1}, q) \leq 1 - \kappa$

and $\varnothing_A(m', n^{-1}, q) \leq 1 - \kappa$ or $\theta_A(mn^{-1}, q) < \kappa$ i.e., $\varnothing_A(mn^{-1}, q) \leq 1 - \kappa$ and $\varnothing_A(m', n^{-1}, q) \leq 1 - \kappa$.

In First Part:

$\left\{ \theta_A(mn^{-1}, q) \right\} \cap \kappa = \kappa$ and $\varnothing_A(mn^{-1}, q) \cap 1 - \kappa = 1 - \kappa$

and so $A^k_m(n, q) = (\kappa, 1 - \kappa)$ and also

$\Rightarrow \theta_A(m, n^{-1}, q) \cap \kappa = \kappa$ and $\varnothing_A(m, n^{-1}, q) \cap 1 - \kappa = 1 - \kappa$

and so

$\Rightarrow A^k_m(n, q) = (\kappa, 1 - \kappa)$. Thus $A^k_m(n, q) = A^k_{m'}(n, q), \forall n \in H$ and $q \in Q$.

$A^k_m = A^k_{m'}$.

In Second Part:

$\left\{ \theta_A(mn^{-1}, q) \right\} \cap \kappa = \theta_A(mn^{-1}, q) < \kappa$

and $\varnothing_A(mn^{-1}, q) \cap 1 - \kappa = 1 - \kappa$

and also

$\Rightarrow \theta_A(m', n^{-1}, q) \cap \kappa = \theta_A(m', n, q) < \kappa$
and $\{0_A(m^{-1}n^{-1}, q) \forall 1 - \kappa \} = 1 - \kappa$

Now since $Nm = Nm'$, therefore let $m = Nm'$, where $a \in N$ and $q \in Q$.

So that $\theta_A(a, q) \geq \kappa$ and $\theta_A(a, q) \leq 1 - \kappa$

$A^\kappa_m(n, q) = ([\theta_A(nm^{-1}, a) \Lambda \kappa, \{\theta_A(nm^{-1}, q) \forall 1 - \kappa\})$

$= (\theta_A(nm^{-1}, a), 1 - \kappa)$

$\geq (\theta_A(a, q) \Lambda \kappa, \{\theta_A(nm^{-1}, q) \forall 1 - \kappa\})$

$= (\theta_A(a, q), 1 - \kappa)$

$= (\{\theta_A(nm^{-1}, q) \Lambda \kappa, \{\theta_A(nm^{-1}, q) \forall 1 - \kappa\})$

Thus $A^\kappa_m(n, q) \geq A^\kappa_m(n, q), \forall n \in H and q \in Q$.

Similarly $A^\kappa_m(n, q) = ([\theta_A(nm^{-1}, a) \Lambda \kappa, \{\theta_A(nm^{-1}, q) \forall 1 - \kappa\})$

$= (\theta_A(nm^{-1}, a), 1 - \kappa)$

$\geq (\theta_A(a, q) \Lambda \kappa, \{\theta_A(nm^{-1}, q) \forall 1 - \kappa\})$

$= (\theta_A(a, q), 1 - \kappa)$

$= (\{\theta_A(nm^{-1}, a) \Lambda \kappa, \{\theta_A(nm^{-1}, q) \forall 1 - \kappa\})$

Thus $A^\kappa_m(n, q) \geq A^\kappa_m'(n, q), \forall n \in H and q \in Q$.

$\therefore A^\kappa_m(n, q) = A^\kappa_m'$.

3.4 Proposition

The set $Q$ and $S$ of all $\kappa$-Intuitionistic $q -$ fuzzy cosets of an $IQFNSG$ $A$ of a group $H and q \in Q$, from a group the well-defined operations $\otimes$.

**Proof:**

It is easy to check that the identity element of $S$ is $A^\kappa_e$, where $e$ is the identity element of group $H$ and , and the inverse of an element $A^\kappa_m$ is $A^\kappa_m^{-1}$.

3.5 Proposition

A mapping $f : G \rightarrow S$, where $G$ is a group and $S$ is the set of all $\kappa$-Intuitionistic $q -$ fuzzy cosets of the $IQFNSGA$ of $G$ defined by $f(m, q) = A^\kappa_m$, is an onto homomorphism with $\ker f = N(= C_1 - \kappa(A, q))$, where $\kappa \in [0,1] and q \in Q$.

**Proof:**

Clearly $f$ is an onto homomorphism

Let $m \in \ker f and q \in Q$, then $f(m, q) = \text{identity element of } S = A^\kappa_e$

Therefore $A^\kappa_m = A^\kappa_e$ so $Nm = Ne = N \Rightarrow m \in N$

$\Rightarrow \ker f \subseteq N$

Conversely, let $m \in N \Rightarrow Nm = N$ so that
\[ Nm^{-1} = Ng^{-1} \forall \ g \in G \text{ and } q \in Q. \]

If possible let \( m \notin \ker f \)
\text{i.e., } \( A^\kappa_m \neq A^\kappa_e \) there fore there exists \( g \in G \text{ and } q \in Q \)
such that \( A^\kappa_m (g, q) \neq A^\kappa_e (g, q) \).

Suppose \( \theta_A (mg^{-1}, q) < \kappa \text{ and } \theta_A (g^{-1}, q) \geq \kappa \), i.e., \( \varnothing_A (mg^{-1}, q) \leq 1 - \kappa \)
and 
\[ \varnothing_A (g^{-1}, q) \leq 1 - \kappa \]

Therefore \( \theta_A (g^{-1}, q) \geq \kappa \) and \( \varnothing_A (g^{-1}, q) \leq 1 - \kappa \Rightarrow g^{-1} \in N \)
so \( Ng^{-1} = N \)
\[ \therefore Ng^{-1} = N \Rightarrow mg^{-1} \in N \text{ and so } \theta_A (mg^{-1}) \geq \kappa, \text{ a contradiction}. \]

Similarly, \( \theta_A (mg^{-1}, q) \geq \kappa \) and \( \theta_A (g^{-1}, q) < \kappa \) is not possible.

\[ \therefore \text{either } \theta_A (mg^{-1}, q) \geq \kappa, \theta_A (g^{-1}, q) < \kappa \text{ i.e., } \varnothing_A (mg^{-1}, q) \leq 1 - \kappa \]
and 
\[ \varnothing_A (g^{-1}, q) \leq 1 - \kappa \]

or
\[ \theta_A (mg^{-1}, q) < \kappa, \theta_A (g^{-1}, q) < \kappa \text{ i.e., } \varnothing_A (mg^{-1}, q) \leq 1 - \kappa \]
and 
\[ \varnothing_A (g^{-1}, q) \leq 1 - \kappa. \]

\textbf{In First Part:}
\[ \{ \theta_A (mg^{-1}, q) \land \kappa \} = \kappa \text{and} \{ \varnothing_A (mg^{-1}, q) \lor 1 - \kappa \} = 1 - \kappa \]
and so \( A^\kappa_m (g, q) = (\kappa, 1 - \kappa) \)

Similarly we get, \( A^\kappa_m (g, q) = (\kappa, 1 - \kappa) \). Thus \( A^\kappa_m (g, q) = A^\kappa_e (g, q) \).

\textbf{In Second Part:}
\[ \{ \theta_A (mg^{-1}, q) \land \kappa \} = \theta_A (mg^{-1}, q) < \kappa \]
and \( \varnothing_A (mg^{-1}, q) \lor 1 - \kappa \) = \( 1 - \kappa \)
\[ A^\kappa_m (g, q) = (\{ \theta_A (mg^{-1}, q) \land \kappa \}, \{ \varnothing_A (mg^{-1}, q) \lor 1 - \kappa \}) \]
\[ = (\theta_A (mg^{-1}, q), 1 - \kappa) \]
\[ \geq (\theta_A (m, q) \land \theta_A (g, q), 1 - \kappa) \]
\[ = (\theta_A (g, q), 1 - \kappa) \quad \therefore m \in N \text{ and } q \in Q \quad \therefore \theta_A (m, q) \geq \kappa \]
and 
\[ \theta_A (g, q) = \theta_A (g^{-1}, q) < \kappa \]
\[ = (\{ \theta_A (eg, q) \land \kappa \}, \{ \varnothing_A (eg^{-1}, q) \lor 1 - \kappa \}) \]
\[ = A^\kappa_e (g, q) \]
\[ = (\theta_A (g^{-1}, q), 1 - \kappa) \]
\[ \geq (\theta_A (mg^{-1}m^{-1}, q), 1 - \kappa) \]
\[ \text{[As A is IQFNSG of G so } \theta_A (mg^{-1}m^{-1}, q) = \theta_A (g^{-1}, q) \] \]
\[ \geq (\theta_A (mg^{-1}, q) \land \theta_A (m, q), 1 - \kappa) \]
\[ = (\theta_A (mg^{-1}, q), 1 - \kappa) \]
\[ = (\theta_A (g^{-1}, q), 1 - \kappa) \]
\[ = A^\kappa_m (g, q) \]
\[ \text{i.e., } f (m, q) = \text{identity element of S and so } m \in \ker f \text{ and } q \in Q. \]
\[ N \subseteq \ker f \text{ker} f = N. \]

\textbf{3.6 Proposition}

If \( f: G \to S \) is an onto homomorphism, then \( f (A, q) = B \), where A is IQFS of G and B is IQFS of S.

\textbf{Proof:}
Let \( A^\kappa_x \in S \) be any element of S, where \( x \in H \text{ and } q \in Q \) such that 
\[ f(x, q) = A^\kappa_x \]
Let A be IQFS of G, then
\[ f(A, q)(A^k_e) = \left\{ \left( \sup \{ \theta_A(m, q) : m \in f^{-1}(A^k_e) \}, \inf \{ \varnothing_A(m, q) : m \in f^{-1}(A^k_e) \} \right) \right\} \]
\[ (\theta_A(e, q), \varnothing_A(e, q)) \]
\[ = \left\{ \left( \sup \{ \theta_A(m, q) : A^k_m = A^k_e \}, \inf \{ \varnothing_A(m, q) : A^k_m = A^k_e \} \right) \right\} \]
\[ (\theta_A(e, q), \varnothing_A(e, q)) \]
\[ = B(A^k_e) \]
Hence \( f(A, q) = B \)

3.7 Theorem

Let A be a IQFNSG of G and B be a IQFSG of S, then \( C_{\kappa,1-\kappa}(B, q) = \{ A^k_e \} \).

Proof:
Now \( B(A^k_e) = \{ \theta_B(A^k_e), \varnothing_B(A^k_e) \}, \)
where \( \theta_B(A^k_e) = \sup \{ \theta_A(m, q) : Nm = N \} \)
\[ = \sup \{ \theta_A(m, q) : m \in N \} \]
\[ \geq \theta_A(a, q), \forall a \in N \text{ and } q \in Q = C_{\kappa,1-\kappa}(A, q) \]
\[ \geq \kappa \]
Similarly, we can show that \( \varnothing_A(A^k_e) \leq 1 - \kappa. \)
Thus \( A^k_e \in C_{\kappa,1-\kappa}(B, q) \)
Let \( A^k_e \in C_{\kappa,1-\kappa}(B, q) \Rightarrow \theta_B(A^k_e) \geq \kappa \) and \( \varnothing_B(A^k_e) \leq 1 - \kappa \)
Let \( \alpha_1 = \theta_B(A^k_e) = \sup \{ \theta_A(m, q) : Nm = Nx \} \)
and \( \alpha_2 = \varnothing_B(A^k_e) = \inf \{ \varnothing_A(m, q) : Nm = Nx \}. \)
Therefore \( \alpha_1 \geq \kappa \) and \( \alpha_2 \leq 1 - \kappa. \)
Let \( \varepsilon > 0 \) be given there exist such that \( m, n \in \text{Hand } q \in Q \) such that
\( Nm = Nx \) so that \( mx^{-1} = a_1 \in N \) and \( \theta_A(m, q) > \alpha_1 - \varepsilon \geq \kappa - \varepsilon \) and
\( Nn = Nx \) so that \( nx^{-1} = a_2 \in N \) and \( \varnothing_A(n, q) < \alpha_2 + \varepsilon \leq (1 - \kappa) + \varepsilon \)
\( \theta_A(x, q) = \theta_A(xa_1a_2^{-1}, q) \geq \theta_A(xa_1, q) \cap \varnothing_A(a_1, q) \)
\[ = \left\{ \begin{array}{ll}
\geq \kappa \text{ if } \theta_A(xa_1, q) \geq \kappa \text{ and } \\
\leq 1 - \kappa \text{ if } \theta_A(xa_2, q) \leq 1 - \kappa 
\end{array} \right\} \]
\( \varnothing_A(x, q) = \varnothing_A(xa_2a_2^{-1}, q) \leq \varnothing_A(xa_2, q) \cup \varnothing_A(a_2, q) \)
\[ = \left\{ \begin{array}{ll}
\leq 1 - \kappa \text{ if } \varnothing_A(xa_2, q) \leq 1 - \kappa \\
\geq \kappa \text{ if } \varnothing_A(xa_2, q) > 1 - \kappa
\end{array} \right\} \]
Thus in any case \( \theta_A(x, q) > \kappa - \varepsilon \) and \( \varnothing_A(x, q) < (1 - \kappa) + \varepsilon, \) for all \( \varepsilon > 0 \)
\[ \Rightarrow \theta_A(x, q) \geq \kappa \text{ and } \varnothing_A(x, q) \leq 1 - \kappa \Rightarrow x \in C_{\kappa,1-\kappa}(A) \]
\[ \Rightarrow Nx = NsoA^k_e = A^k_e \]
Hence \( C_{\kappa,1-\kappa}(B, q) = \{ A^k_e \}. \)

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4. References

4.1. Journal Article

