A Transformation of the Navier-Stokes Equations for Efficient Fluid Animation

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Abstract

Computational efficiency is an important theme in fluid simulation since its heavy computational load is burden to designers. Our objective is to proposed a grid based method satisfying divergence free property without the projection method. The paper transforms the momentum conservation part of the Navier-Stokes equations to vector potential based equations. Our method do not need any global operations for mass conservation. The transformed equation is solved numerically in similar manner to fractional step method.

Keywords Computer graphics, Physics based animation, Fluid simulation, Projection method

1 Motivation

Grid based approach to numerically solve the Navier-Stokes equations is most popular approach for producing realistic fluid animation. Projection step is the worst bottleneck in the grid based fractional step method. This bottleneck is due to global operation characteristic of the projection step. Our objective is to proposed a grid based method satisfying divergence free property without the projection method. Computational efficiency is an important theme in fluid simulation since its heavy computational load is burden to designers.

The notorious projection problem has been interested in many researchers. One group tried to reduce computation time itself for solving the linear system of extremely high dimension. GPU based algorithms and advanced numerical analysis methods are located in the group. The other group investigated local operation method such as SPH [2] and LBM [3] in spite of their quality degradation. Our method solve this problem thoroughly, and suggest novel paradigm in fluid simulation.

2 Related work

Many researchers have proposed efficient algorithms for linear system brought by incompressibility of fluid: GPU instead of CPU, preconditioned CGM [5] and Multigrid instead of general CGM. Such methods improved the computational efficiency surprisingly. However, they did not remove the large scale linear system completely.
SPH (Smoothed Particle Hydrodynamics) is a relatively recent approach to enhance the computation load of fluid simulation. Basic SPH consists of only local operations, and are hard to satisfy incompressibility. Some improved SPH were proposed to match the incompressibility with high computational load.

3 Overview and Contribution

We change the momentum conservation part of the Navier-Stokes equations [1] whose variables are velocity and pressure to an equations whose variable is potential field. Here, the potential field has a characteristic that velocity is the curl of the potential field. Since curl of arbitrary vector field is divergence zero, we don’t need a process for mass conservation, e.g. projection step in “Stable fluid”. The transformed equation is solved numerically in similar manner to fractional step method. Here, since there is not any global operation, we gain the advance of computation speed.

Computational load of our method is almost same to the fractional step method except for projection step. As experimental examples show, quality loss is insignificant. Our method solves the transformed Navier-Stokes equations. Theoretically, we remove the need of projection step.

4 Governing equation

The Navier-Stokes equations for invicid incompressible fluid:

\[ u_t + (u \cdot \nabla)u = -\nabla p + f \]
\[ \nabla \cdot u = 0 \]  
(1)  
(2)

The potential version of the Navier-Stokes equation is:

\[ \nabla \times^{-1} [u_t + (u \cdot \nabla)u] = -\nabla p + f \]  
(3)

The potential field \( \Psi \) is defined as

\[ \nabla \times \Psi = u. \]

(4)

Then,

\[ \nabla \times^{-1} u_t = \Psi_t \]

(5)

Eq.3 becomes

\[ \Psi_t + \nabla \times^{-1} [(u \cdot \nabla)u] = \nabla \times^{-1} (\nabla p) + \nabla \times^{-1} f + \nabla q \]

(6)

Inverse curl operator \( \nabla \times^{-1} \) is not one-to-one. For a vector potential \( \mathbf{V} \) and any continuously differentiable scalar function \( q \), \( \nabla \times (\mathbf{V} + \nabla q) = \nabla \times \mathbf{V} \) since \( \nabla \times \nabla q = 0 \).

Then, Eq.6 can be formulated by the fractional step method with given \( \Psi_n \) at time \( n \):

\[ \frac{\Psi_1 - \Psi_n}{\Delta t} = \nabla \times^{-1} f \]

(7)

\[ \frac{\Psi_2 - \Psi_1}{\Delta t} + \nabla \times^{-1} [(u \cdot \nabla)u] = 0 \]

(8)

\[ \frac{\Psi_{n+1} - \Psi_2}{\Delta t} = \nabla \times^{-1} (\nabla p) + \nabla q \]

(9)
4.1 External force

For Eq.7, it is needed to treat global operations for arbitrary forces. However, if gravity is the only external force, one can easily calculate its inverse $\nabla \times -f$, which has a explicit form. Moreover, the inverse can be calculate by using inverse curl operator without global operation as long as the force is given as an analytic function.

4.2 Advection

In this section, we treat advection step (Eq. 8) in the above fractional step method. The problem is how to handle term $\nabla \times -1 [(\mathbf{u} \cdot \nabla)\mathbf{u}]$.

Let’s expand $\nabla \times ((\mathbf{u} \cdot \nabla)\Psi)$:

$$\nabla \times ((\mathbf{u} \cdot \nabla)\Psi) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\mathbf{u} \cdot \nabla)\psi_1 & (\mathbf{u} \cdot \nabla)\psi_2 & (\mathbf{u} \cdot \nabla)\psi_3 \end{vmatrix}$$ (10)

where, $\Psi = (\psi_1, \psi_2, \psi_3)$.

For simplicity, let’s think only $i$-component of Eq. 10:

$$(\mathbf{u} \cdot \nabla)\psi_3 \times y - (\mathbf{u} \cdot \nabla)\psi_2 \times z$$ (11)

where, $(\cdot)_y$ means $\frac{\partial \cdot}{\partial y}$.

Eq. 11 expands into

$$(u_1\psi_3 + u_2\psi_3 + u_3\psi_3) - (u_1\psi_2 + u_2\psi_2 + u_3\psi_2) \cdot z$$ (12)

where, $\mathbf{u} = (u_1, u_2, u_3)$.

The first part of Eq. 12 expands into

$$u_1\psi_3x + u_1\psi_3x + u_2\psi_3y + u_2\psi_3y + u_3\psi_3z + u_3\psi_3z$$ (13)

Similarly, the second part of Eq. 12 expands into

$$u_1\psi_2x + u_1\psi_2x + u_2\psi_2y + u_2\psi_2y + u_3\psi_2z + u_3\psi_2z$$ (14)

We find that $i$-component of $(\mathbf{u} \cdot \nabla)(\nabla \times \Psi)$ consists of the even parts of Eq. 13 and 14. Therefore, we have

$$\nabla \times ((\mathbf{u} \cdot \nabla)\Psi) = (\mathbf{u} \cdot \nabla)(\nabla \times \Psi) + \mathbf{K}$$ (15)

where, $i$-component of $\mathbf{K}$ consists of the odd parts of Eq. 13 and 14. That is,

$$k_1 = (u_1\psi_3x + u_2\psi_3y + u_3\psi_3z) - (u_1\psi_2x + u_2\psi_2y + u_3\psi_2z) \cdot z$$ (16)

$$= \frac{\partial \mathbf{u}}{\partial y} \cdot \nabla \psi_3 - \frac{\partial \mathbf{u}}{\partial z} \cdot \nabla \psi_2$$

where, $\mathbf{K} = (k_1, k_2, k_3)$. 

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Similar calculations from Eq. 11 to Eq. 16 for \( j \) and \( k \)-components of Eq. 10 give \( k_2 \) and \( k_3 \):

\[
k_2 = \frac{\partial u}{\partial z} \cdot \nabla \psi_1 - \frac{\partial u}{\partial x} \cdot \nabla \psi_3 \quad (17)
\]

\[
k_3 = \frac{\partial u}{\partial x} \cdot \nabla \psi_2 - \frac{\partial u}{\partial y} \cdot \nabla \psi_1 \quad (18)
\]

Eqs. 16, 17, and 18 can be summarized as:

\[
\mathbf{K} = \begin{vmatrix}
i & j & k \\
\psi_1 & \psi_2 & \psi_3
\end{vmatrix} \quad (19)
\]

Let’s assume

\[
\mathbf{K} = \nabla \times \mathbf{A} \quad (20)
\]

where, \( \mathbf{A} = (a_1, a_2, a_3) \).

By Eqs. 4 and 20, Eq. 15 can be rewritten as:

\[
\nabla \times \left((u \cdot \nabla)\Psi\right) = (u \cdot \nabla)u + \nabla \times \mathbf{A} \quad (21)
\]

When applying inverse curl \( \nabla \times^{-1} \) to Eq. 21, we get

\[
(u \cdot \nabla)\Psi = \nabla \times^{-1} [(u \cdot \nabla)u] + \mathbf{A} \quad (22)
\]

\[
\nabla \times^{-1} [(u \cdot \nabla)u] = (u \cdot \nabla)\Psi - \mathbf{A} \quad (23)
\]

By using Eq. 23, Eq. 8 can be rewritten as:

\[
\frac{\Psi_2 - \Psi_1}{\Delta t} + (u \cdot \nabla)\Psi - \mathbf{A} = 0 \quad (24)
\]

Eq. 24 can be separated by following two substeps:

\[
\frac{\hat{\Psi}_2 - \Psi_1}{\Delta t} + (u \cdot \nabla)\Psi = 0 \quad (25)
\]

\[
\frac{\Psi_2 - \hat{\Psi}_2}{\Delta t} - \mathbf{A} = 0 \quad (26)
\]

\[
\Rightarrow \Psi_2 = \hat{\Psi}_2 + \Delta t \mathbf{A} \quad (27)
\]

Eq. 25 can be regarded as the advection process for potential field. We have an additional step (Eq. 27) due to the transformation of the Navier-Stokes equations to our potential based form. Here, \( \mathbf{K} \) is a function of intermediate potential \( \hat{\Psi}_2 \).

We observe that it is straightforward to solve inverse curl \( \mathbf{A} \) of \( \mathbf{K} \) in Eq. 20 if we don’t care boundary conditions. Our framework handles boundary conditions in the last step of whole processes.
Figure 1. For efficient numerical formulation, $A_1$ and $A_3$ are defined on the faces parallel to $YZ$ plane, etc.

5 Inverse curl

Given $K = (K_1, K_2, K_3)$, we want to find $A = (A_1, A_2, A_3)$ such that $\nabla \times A = K$ (Figure 1).

\[
\begin{vmatrix}
  i & j & k \\
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
  A_1 & A_2 & A_3
\end{vmatrix} = K_1 i + K_2 j + K_3 k \tag{28}
\]

\[
\Rightarrow \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = K_1 \tag{29}
\]
\[
\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} = K_2 \tag{30}
\]
\[
\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x} = K_3 \tag{31}
\]

6 Conclusion

Our method proposes a grid based method satisfying divergence free property without the projection method. It transforms the momentum conservation part of the Navier-Stokes equations to vector potential based equations. Our method do not need any global operations for mass conservation. We can solve the transformed equation numerically using a similar method to fractional step method.

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References


