Performance Analysis of Best Relay Selection in Cooperative Wireless Networks

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Abstract
Cooperative communication has been proposed recently in wireless communication systems because of its producing high network reliability and considerable error rate decrease with an extension in the coverage area in wireless networks without need for an increase in the infrastructure. The most challenging problem in implementing cooperation protocols in cooperative wireless networks is the relay selection. This paper proposes a relay selection technique based on Decode-and-Forward (DF) relaying protocol, using the available channel state information (CSI) at the source and the relays. We first establish a closed form symbol error rate (SER) by using the concept of moment generating function (MGF) for both M phase shift keying (MPSK) and M quadrature amplitude modulation (MQAM) signals. By finding the cumulative density function (CDF) and the probability density function (PDF), this facilitated the derivation of a new SER and can propose an optimal power method with relay selection (RS) technique where only the best relay is selected that maximizes the signal-to-noise ratio (SNR) to achieve the transmission process and develop the performance of the system. Finally, simulation results provided good confirmation of the theoretical analysis.

Keywords: Cooperative Communications, Decode-and-Forward, Relay Selection, Optimal Power

1. Introduction

Recently, the demands for small and low cost devices have been increasing in emerging wireless applications. The main reasons that motivate focusing on the development of these systems are the limited battery lifetime of devices and the insufficient bandwidth shared by a large number of users. Therefore, many research works is undertaken to maximize the system performance and the effectiveness of these solutions is when users are willing to share their local resources and cooperate in transmitting each other’s messages, which is the essence of cooperative communications [1]. Cooperative communications make use of the spatial diversity in multiuser systems by allowing users with various channel qualities to cooperate and relay each other’s signals to the destination [2]; this is refers to a system where users share and manage their resources to improve the transmission quality and this is generally attractive in wireless environments due to the varied channel quality and the partial energy and bandwidth resources [3, 4].

The wireless channels suffer from fading [5, 6], thus the signal attenuation can vary considerably over a given transmission. Diversity is caused by transmitting the
independent copies of the signal and be capable of combating the harmful effects of fading. By transmitting signals from different locations, spatial diversity is generated and allowing independent faded version of the signal at the receiver. Cooperative communication generates this diversity in a new and interesting way, thus it is proposed recently with the advantage of broadcast in wireless medium which permit several nodes cooperatively transmit signals to a destination together. Many researchers have shown that cooperative communication can present important performance improvements in terms of increased capacity, improved transmission reliability, spatial diversity and diversity-multiplexing tradeoff [7, 8].

The performance of the wireless network can be improved by using of multiple-relay networks [9]. In this work, we suggest a cooperative protocol based on the best relay selection technique with the availability of the channel state information (CSI) at the source and the relays. Various cooperative wireless networks were proposed and analyzed that used the relay selection technique in [10-14], where the authors assumed that the power is equal between the source and the selected relays. In [15-17], the authors use relay selection and calculate the outage probability and channel capacity for the cooperative model. Within this paper, relay selection (RS) is employed in the Decode-and-Forward (DF) cooperative wireless network model in order to get developments in the multi-relay network, where only two time slots are necessary (one for the direct link and one for the best relay selected) to assist the transmission process.

The rest of the paper is organized as follows. In Section 2, we describe the DF cooperative model for multi-node wireless networks analyzed under Rayleigh fading channels. In Section 3, we analyze the SER performance by using the concept of moment generating function (MGF) and obtained the exact SER expression under Rayleigh fading channels and explaining it with two types of modulation signals: M phase shift keying (MPSK) and M quadrature amplitude modulation (MQAM) modulation. In section 4, we determine SER with the best relay selected according to its SNR value. Section 5 contains the analysis for the SER with relay selection by improving the optimal power technique. The simulation results are presented in Section 6. Finally Section 7 presents the conclusions from the work in the paper.

2. System Model

Figure 1 represents the DF protocol of wireless cooperative network consisting of source (S), multi-relay (R1,R2,…,Rn) and destination (D). The system is work under Rayleigh fading channel and the noise assumed to be additive white Gaussian noise (AWGN). The main channel gains for the transmitter are known, and assuming that the receiver has the perfect CSI. By using the multiple access technique TDMA, FDMA or CDMA scheme, users transmitted signals through orthogonal channels. The DF cooperative wireless network, considered here, is involved in two phases.
In the first phase, the source transmits the symbol and then received from the destination and all relays owing to the broadcast feature of the wireless channel. During the first phase, the signals received at the destination and relays are respectively written as:

\[ Y_{SD} = \sqrt{P_S} h_{SD} x + n_{SD} \]  
\[ Y_{SR_i} = \sqrt{P_S} h_{SR_i} x + n_{SR_i} \]

where \( x \) is the transmitted information symbol from the source, \( Y_{jk} \) illustrate the received signal from node \( j \) to node \( k \), \( h_{jk} \) are the fading channel coefficients from node \( j \) to node \( k \), and \( n_{jk} \) are the corresponding AWGN with variance \( N_0 \) from node \( j \) to node \( k \).

During the second phase, the source and the relays transmit signal to the destination. In this phase, the relays correctly decode the received signal and the received signal at the destination from the relays can be written as follows:

\[ Y_{R_iD} = \sqrt{P_{R_i}} h_{R_iD} x + n_{R_iD} \]
Where $\bar{x}$ is the decoded information at the $i^{th}$ relay, $P_S$ and $P_{R_i}$ are the power at the source and at the $i^{th}$ relay respectively. From these expressions, we derive the following relations:

$$\gamma_S = P_S \frac{|h_{SD}|^2}{N_0}$$

(4)

$$\gamma_{R_i} = P_{R_i} \frac{|h_{R_iD}|^2}{N_0}$$

(5)

Where $\gamma_S$ and $\gamma_{R_i}$ are the instantaneous SNR of the direct and the $i^{th}$ relay respectively. The variable $h_{jk}$ which defined before, is modeled as the independent zero-mean, circularly-symmetric complex Gaussian random variable with variance one. By using the maximum ratio combiner (MRC) [18, 19], the signals from source and $i^{th}$ relay are combined at the destination and the received SNR is:

$$\gamma = \gamma_S + \sum_{i=1}^{N} \gamma_{R_i}$$

(6)

\(\gamma_S\) and \(\gamma_{R_i}\) follow exponential distribution with parameters:

$$\lambda_S = \frac{N_0}{P_S \sigma_{SD}^2}$$

(7)

$$\lambda_{R_i} = \frac{N_0}{P_{R_i} \sigma_{R_iD}^2}$$

(8)

Where $\sigma_{SD}^2$ and $\sigma_{R_iD}^2$ are the variance of $h_{SD}$ and $h_{R_iD}$ respectively.

3. SER Performance Analysis

In this section, we analyze the SER performances of multi-relay system to evaluate decode-and-forward transmission for both MPSK and MQAM modulation signals. The conditional SER with SNR $\gamma$ is described in [20], for MPSK signals as follows:

$$P_{MPSK} = \frac{1}{\pi} \int_0^{\pi} \exp\left(-\frac{by}{\sin^2\theta}\right) d\theta$$

(9)

And the conditional SER in relation to SNR $\gamma$ with MQAM signals can be similarly determined as:

$$P_{MQAM} = \frac{4K}{\pi} \int_0^{\pi} \exp\left(-\frac{by}{2\sin^2\theta}\right) d\theta - \frac{4K^2}{\pi} \int_0^{\pi} \exp\left(-\frac{by}{2\sin^2\theta}\right) d\theta$$

(10)

By averaging the conditional SER over the allocation of $\gamma_S$ and $\gamma_{R_i}$, then the unconditional SER for MPSK and MQAM signals of the proposed system are given as:

$$P_{MPSK} = \frac{1}{\pi} \int_0^{\pi} \frac{M_{\gamma_S}(-t) \prod_{i=1}^{N} M_{\gamma_{R_i}}(-t)}{\gamma_S}$$

(11)
\[ P_{MQAM} = \left[ \frac{4K}{\pi} \int_{0}^{\frac{\pi}{2}} - \frac{4K^2}{\pi} \int_{0}^{\frac{\pi}{2}} \right] M_{\gamma_S}(-t) \prod_{i=1}^{n} M_{\gamma_{R_i}}(-t) \, d\theta \]  

Where \( M_{\gamma_S}(-t) \) and \( M_{\gamma_{R_i}}(-t) \) are the MGF's of \( \gamma_S \) and \( \gamma_{R_i} \) respectively and the values of \( b \) and \( t \) are as given in Table 1. Substituting the moment generating function's expression, then equations (11) and (12) can be given as:

\[ P_{MPSK} = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} \left( -\frac{\sin^2\theta}{\sin^2\theta + \frac{b}{\lambda_S}} \right) \prod_{i=1}^{n} \left( -\frac{\sin^2\theta}{\sin^2\theta + \frac{b}{\lambda_{R_i}}} \right) \, d\theta \]  

\[ P_{MQAM} = \left( \frac{4K}{\pi} \int_{0}^{\frac{\pi}{2}} - \frac{4K^2}{\pi} \int_{0}^{\frac{\pi}{2}} \right) \prod_{i=1}^{n} \left( -\frac{2\sin^2\theta}{2\sin^2\theta + \frac{b}{\lambda_S}} \right) \, d\theta \]

Equations (13) and (14) represent the exact expressions for the SER for both MPSK and MQAM respectively of the proposed system. This expression is not pliable in analysis, so we derive a SER lower bound that is converges to the same limit as a theoretical SER upper bound to apply performance analysis. We set up a tight SER lower bound using the truth that \( 0 \leq \sin^2\theta \leq 1 \) as following:

\[ A \left( \frac{1}{1+(b/\lambda_S)} \right) \prod_{i=1}^{n} \left( \frac{1}{1+(b/\lambda_{R_i})} \right) \leq P_{SER} \leq A \left( \frac{\lambda_S}{b} \right) \prod_{i=1}^{n} \left( \frac{\lambda_{R_i}}{b} \right) \]

Where \( A = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} n! \frac{\sin^{2n+2}}{b^{n+1}} \, d\theta \) in MPSK, and \( A = \left( \frac{4K}{\pi} \int_{0}^{\frac{\pi}{2}} - \frac{4K^2}{\pi} \int_{0}^{\frac{\pi}{2}} \right) n! \frac{2\sin^{2n+2}}{b^{n+1}} \, d\theta \) in MQAM. Proof is presented in the appendix.

**Table 1. Simulation Parameters of the Proposed Model**

<table>
<thead>
<tr>
<th>DF parameters</th>
<th>Modulation technique</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>MPSK</td>
</tr>
<tr>
<td>( b )</td>
<td>( \sin^2(\pi/M) )</td>
</tr>
<tr>
<td>( t )</td>
<td>( b/\sin^2\theta )</td>
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**4. Performance of the SER with Best Relay Selection Scheme**

This section presents the expression of the SER with the best relay that is selected as the relay supply the maximum received SNR as shown in Figure 2.
The following relations are acquired in order to obtain the best relay termed as $R_{sel}$.

$$R_{sel} = \arg \{ \max_{i=1,2,...,n} \{ \gamma_{R_i} \} \}$$  \hspace{1cm} (16)

From relation (16), we can obviously know that the relay with the maximum SNR can be selected to take part in the communication process. Mathematically, the instantaneous SNR is given for the best relay selection as:

$$\gamma_{sel} = \max_{i=1,2,...,n} \{ \gamma_{R_i} \}$$  \hspace{1cm} (17)

We can set the analytical expressions for the CDF of $\gamma_{sel}$ with the assumption that all channels are independent and distributed identically:

$$F_{\gamma_{sel}}(\gamma) = P \{ \max_{i=1,2,...,n} \{ \gamma_{R_i} \} < \gamma \} = \prod_{i=1}^{n} P(\gamma_{R_i} < \gamma) = (F_{\gamma_{R_i}}(\gamma))^n$$  \hspace{1cm} (18)

Where $F_{\gamma_{R_i}}(\gamma)$ is the CDF of $\gamma_{R_i}$, from the suggested design, we have $\gamma_{R_i}$ as exponential random variable with $\lambda_{R_i}$, then the CDF can be written as:

$$F_{\gamma_{R_i}}(\gamma) = 1 - e^{-\lambda_{R_i}\gamma}$$  \hspace{1cm} (19)
Now, substituting (19) in (18), then we can get:

$$F_{\gamma_{sel}}(\gamma) = (1 - e^{-\lambda_{Ri} \gamma})^n = \sum_{j=0}^{n} C_j^n (-1)^j e^{-j\lambda_{Ri} \gamma}$$  \hspace{1cm} (20)

Where $C_j^n = (n!/j!(n-j)!)$ is the binomial coefficient, from relation (20), the PDF of $\gamma_{sel}$ can be expressed as the derivative of the CDF with respect to $\gamma$, then the PDF of $\gamma_{sel}$ can be written as:

$$f_{\gamma_{sel}}(\gamma) = \sum_{j=1}^{n} C_j^n j(-1)^{j-1} \lambda_{Ri} e^{-j\lambda_{Ri} \gamma}$$  \hspace{1cm} (21)

With (21), the MGF of $\gamma_{sel}$ can be defined by:

$$M_{\gamma_{sel}}(u) = \int_0^{\infty} e^{-uy} f_{\gamma_{sel}}(y) \, dy$$  \hspace{1cm} (22)

Now, we can get the expression for the MGF of $\gamma_{sel}$ by substituting of (21) in (22) and solving the integral as given below;

$$M_{\gamma_{sel}}(u) = \sum_{j=1}^{n} C_j^n j(-1)^{j-1} \frac{\lambda_{Ri}}{u+j\lambda_{Ri}}$$  \hspace{1cm} (23)

The SER of the relay selection using both MPSK and MQAM signals can be given as:

$$P_{MPSK,S} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_{sel}}(-\frac{b}{\sin^2 \theta}) M_{\gamma_{sel}}(-\frac{b}{2\sin^2 \theta}) \, d\theta$$  \hspace{1cm} (24)

$$P_{MQAM,S} = \left[ \frac{4K}{\pi} \int_0^{\frac{\pi}{2}} - \frac{4K^2}{\pi} \int_0^{\pi/2} \, M_{\gamma_{sel}}(-\frac{b}{2\sin^2 \theta}) M_{\gamma_{sel}}(-\frac{b}{2\sin^2 \theta}) \, d\theta \right]$$  \hspace{1cm} (25)

We substitute the MGF with its values, then relations (24) and (25) becomes as:

$$P_{MPSK,S} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} C_j^n j(-1)^{j-1} \left(-\frac{\sin^2 \theta}{j\sin^2 \theta + \frac{b}{\lambda_{Ri}}}\right) \left(-\frac{\sin^2 \theta}{2\sin^2 \theta + \frac{b}{\lambda_{Ri}}}\right) \, d\theta$$  \hspace{1cm} (26)

$$P_{MQAM,S} = \left[ \frac{4K}{\pi} \int_0^{\frac{\pi}{2}} - \frac{4K^2}{\pi} \int_0^{\pi/2} C_j^n j(-1)^{j-1} \left(-\frac{2\sin^2 \theta}{2\sin^2 \theta + \frac{b}{\lambda_{Ri}}}\right) \left(-\frac{2\sin^2 \theta}{2\sin^2 \theta + \frac{b}{\lambda_{Ri}}}\right) \, d\theta \right]$$  \hspace{1cm} (27)

The SER obtained in (26) and (27) can be decomposed as follows:

$$P_{SERS} = P_{odd} - P_{even}$$  \hspace{1cm} (28)

$P_{odd}$ and $P_{even}$ for MPSK and MQAM are as given below:

$$P_{odd,MPSK} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} C_{2j-1} C_{2j} \left(-\frac{\sin^2 \theta}{(2j-1)\sin^2 \theta + \frac{b}{\lambda_{Ri}}}\right) \left(-\frac{\sin^2 \theta}{\sin^2 \theta + \frac{b}{\lambda_{Ri}}}\right) \, d\theta$$  \hspace{1cm} (29)
\[ P_{\text{even},\text{MPSK}} = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} \sum_{j=1}^{q} C_{2j}^{n} 2j \left( -\frac{\sin^2\theta}{2\sin^2\theta + \frac{b}{\lambda_{R_i}}} \right) \left( -\frac{\sin^2\theta}{\sin^2\theta + \frac{b}{\lambda_{S}}} \right) d\theta \] (30)

\[ P_{\text{odd},\text{MQAM}} = \left( \frac{4K}{\pi} \int_{0}^{\frac{\pi}{2}} - \frac{4K^{2}}{\pi} \int_{0}^{\frac{\pi}{2}} \sum_{j=1}^{q} C_{2j-1}^{n} (2j - 1) \left( -\frac{2\sin^2\theta}{2(2j-1)\sin^2\theta + \frac{b}{\lambda_{R_i}}} \right) \left( -\frac{2\sin^2\theta}{2\sin^2\theta + \frac{b}{\lambda_{S}}} \right) d\theta \right) \] (31)

\[ P_{\text{even},\text{MQAM}} = \left( \frac{4K}{\pi} \int_{0}^{\frac{\pi}{2}} - \frac{4K^{2}}{\pi} \int_{0}^{\frac{\pi}{2}} \sum_{j=1}^{q} C_{2j}^{n} 2j \left( -\frac{2\sin^2\theta}{4\sin^2\theta + \frac{b}{\lambda_{R_i}}} \right) \left( -\frac{2\sin^2\theta}{2\sin^2\theta + \frac{b}{\lambda_{S}}} \right) d\theta \right) \] (32)

5. Optimal Power with Best Relay Selection

In this section, we adjust some analysis for the SER as of relay selection by improving optimal power technique in order to develop the performance of the design model by reducing the value of the SER. We examine the optimal powers at the source and the best relay selected and solving such optimization problem to find the asymptotic performance of the system that gives useful approach in the analysis. For achieving this approach, we must execute some analysis on the PDF and the CDF depending on the expression of the SER with the MGF. From relation (20), the expression of the CDF can be approximated as \( F_{y_{\text{sel}}} (y) \approx (\lambda_{R_i})^{\gamma} \) and the PDF become as \( f_{y_{\text{sel}}} (y) \approx n(\lambda_{R_i})^{n-1} (y)^{n-1} \) with the MGF is \( M_{y_{\text{sel}}}(t) = \int_{0}^{\infty} n(\lambda_{R_i})^{n} e^{-ty} (y)^{n-1}dy \). Now, we define the following terms:

\[ I_{n} = \int_{0}^{\infty} n(\lambda_{R_i})^{n} e^{-ty} y^{n-1}dy \] (33)

From the above equation, we can appreciate that \( I_{n+1} = \frac{n+1}{t} I_{n} \). After applying the analysis with the MGF, we get \( I_{n} = \frac{n}{t} \approx M_{y_{\text{sel}}}(t) \). Now introduce the approximate expression of the MGF for \( y_{S} \) and \( y_{\text{sel}} \) into (26) and (27), then we get the tractable expression for both MPSK and MQAM signals as:

\[ P_{\text{app},\text{MPSK}} = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} n! \lambda_{S}(\lambda_{R_i})^{n} \frac{\sin^{2n+2\theta}}{b^{n+1}} d\theta \] (34)

\[ P_{\text{app},\text{MQAM}} = \left( \frac{4K}{\pi} \int_{0}^{\frac{\pi}{2}} - \frac{4K^{2}}{\pi} \int_{0}^{\frac{\pi}{2}} n! \lambda_{S}(\lambda_{R_i})^{n} \frac{2\sin^{2n+2\theta}}{b^{n+1}} d\theta \right) \] (35)

Now we can make an optimization problem based on relations (34) and (35) by substituting \( \lambda_{S} \) and \( \lambda_{R_i} \) with their values:

\[ \left\{ \begin{array}{l} \min_{(P_{S},P_{R})} \left( A \frac{1}{P_{S}} \prod_{i=1}^{n} \frac{1}{P_{R}} \right) \\ P_{S} + P_{R} = P \end{array} \right. \] (36)

By applying the logarithm function to relation (36) of the first term, we obtain:

\[ G(P_{S},P_{R}) = -\ln(P_{S}) + n\ln\left(\frac{1}{P_{R}}\right) \] (37)
Now we take the derivative of (37) with respect to $P_S$, we get:

$$\frac{\partial G}{\partial P_S} = -\frac{1}{P_S} + \frac{n}{P_R}$$

(38)

We can obtain the following relations by setting relation (38) to zero:

$$P_R = nP_S$$

(39)

By taking the relation between $P_S$ and $P_R$, we have:

$$P_S = \frac{1}{n+1} P, \quad P_R = \frac{n}{n+1} P$$

(40)

The above expressions included in relation (40) are the optimal parameters gained by the relay selection. With feedback channel, both the source and the selected relay will be notified by the destination with their power levels with the gained parameters.

6. Simulation Results

In this section the simulation results for both MPSK and MQAM signals are presented to verify the mathematical analysis. We consider that $N_0$ is unit, the variance of the fading channels $\sigma^2_{S,D}, \sigma^2_{R,i,D}$ are both equal to one, and the total power $P$ is to be used as constant power.

Figures 3 and 4, explain the performance of RS, where the source cooperates with the best selected relay amongst the number of relays in the DF cooperative wireless network model. By using RS, we can see that when we increase the numbers of relays, the value of SER decreases and from results we can perceive the tightness of the analytical analysis. We can notice that we put without selection (WS) in order to compare it with the RS to confirm the improvement of the results gained with RS scheme. This result can be considered as an important because its illustrates the large benefit of RS when only two channel links are to be used to aid the performance transmission in cooperative wireless network in place of the contribution of all relay nodes which may cause decrease in the spectral efficiency of the model as a result of increase in the number of channel links.

Figures 5 and 6 show the performance of optimal and equal power ($P_S = P_{R,i} = \frac{P}{2}$) for RS and WS. From these figures, we can see that RS with optimal power is better than WS with optimal power; this result explains the development when using RS in wireless communication system. From the figures, we can also see that RS with optimal power perform is better than RS with equal power, this is because of the distinction of the suggested optimal power technique. We can see also the importance of this result when only two channel links are to be used in the cooperative system, where feedback channel sends from the receiver to inform the source and the best selected relay by their power level that can be gained from relation (40).
Figure 3. SER Representation with and without Relay Selection for MPSK Modulation Signals

Figure 4. SER Representation with and without Relay Selection for MQAM Modulation Signals
Figure 5. SER Comparisons between Power Schemes with and without Relay Selection for MPSK Modulation Signals

Figure 6. SER Comparisons between Power Schemes with and without Relay Selection for MQAM Modulation Signals
7. Conclusions

In this paper, we proposed a cooperative wireless network with DF multi-relay system over Rayleigh fading channel by using relay selection technique. After setting up the expressions of the SNR and SER, we established a tight SER lower bound with both MPSK and MQAM signal. A development for the SER with an optimal power technique by using best relay selection technique was proved in order to enhance the performance of the system. In addition, the derived SER lower bound could be used to determine the optimum transmission power for the source node and the best relay selected then the optimum power allocation could enhance the error probability of the system efficiently, mainly in the low SNR regime. Simulation results show the tightness of the theoretical analysis and the advantage of the proposed scheme in wireless cooperative communication systems.

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Appendix: proof of relation (15)

As we have \(0 \leq \sin^2 \theta \leq 1\), we can derive the following inequalities:

\[
0 \leq \sin^2 \theta \leq 1 \quad \Rightarrow \quad \frac{b}{\lambda_S} \leq \sin^2 \theta + \frac{b}{\lambda_S} \leq 1 + \frac{b}{\lambda_S}
\]

\[
\frac{\sin^2 \theta}{1 + \frac{b}{\lambda_S}} \leq \frac{\sin^2 \theta + \frac{b}{\lambda_S}}{1 + \frac{b}{\lambda_S}} \leq \frac{\sin^2 \theta}{1 + \frac{b}{\lambda_S}}
\]  

(41)

\[
\prod_{i=1}^{n} \left( \frac{\sin^2 \theta}{1 + \lambda_R i} \right) \leq \prod_{i=1}^{n} \left( \frac{\sin^2 \theta + \frac{b}{\lambda_S}}{1 + \lambda_R i} \right) \leq \prod_{i=1}^{n} \left( \frac{\sin^2 \theta}{1 + \lambda_S} \right)
\]

(42)

Multiplying (41) and (42), then we obtain:

\[
\frac{\sin^2 \theta}{1 + \frac{b}{\lambda_S}} \prod_{i=1}^{n} \left( \frac{\sin^2 \theta + \frac{b}{\lambda_S}}{1 + \lambda_R i} \right) \leq \frac{\sin^2 \theta}{1 + \lambda_S} \prod_{i=1}^{n} \left( \frac{\sin^2 \theta + \frac{b}{\lambda_S}}{1 + \lambda_R i} \right) \leq \frac{\sin^2 \theta}{1 + \lambda_S} \prod_{i=1}^{n} \left( \frac{\sin^2 \theta}{1 + \lambda_R i} \right)
\]

(43)

Now subsequently we take the integral of (43), then we prove the relation (15).

References
