Fault Detection for Linear Switched Systems with Average Dwell Time Using Unknown Input Observers

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Abstract

The linear switched system is a kind of hybrid system that simultaneously possesses characteristics of both continuous event systems and discrete event systems. In this paper the issue of fault detection of switched systems is investigated using unknown input observers (UIO) and an average dwell time approach. The key parameter of UIOs is designed by solving linear matrix inequalities, which differs from traditional design methods. Based on UIOs, a residual generator is derived for detecting faults. A simulation case study validates the proposed approach.

Keywords: Switched System, Fault Detection, Unknown Input Observers, Linear Matrix Inequality, Average Dwell Time

1. Introduction

Switched systems are a class of simple and typical hybrid systems, which consist of a family of continuous dynamic subsystems and a switched signal that determines the switching sequence among different subsystems [1]. Therefore, the dynamic behavior of the whole system is established by all of the subsystems and the switched signal. The switched systems usually are used to describe some complex physical systems or processes, and recognized as one of the international frontiers in theoretical research of hybrid system. The stability analysis of switched systems has attracted great attention and obtained many results [2-3]. In addition, the fault detection of this kind of system has also caused the interest of researchers [4-5].

Fault detection is a kind of technology for increasing the reliability and ensuring security of the system. After decades of development, a large number of detecting methods have been proposed [6-7]. Among all of these approaches, it is known that the state observers [8-10] are the most common means which can produce residual for the purpose of detecting faults.

Unknown input observers (UIOs) are special state observers, which can achieve perfect decoupling unknown inputs from the residuals and realize accurate state estimation. In [11], the faults of linear system are isolated by using UIOs. In [12], a sort of UIOs for vector second order system is proposed and applied to detect fault. In [13], the UIOs of nonlinear system are designed and given in the form of linear matrix inequality (LMI). To our knowledge, the fault detection of switched system with average dwell time using UIOs is not yet developed and worthy of research. For the purpose, we would focus on the problem in this paper. Firstly the parameters of UIOs are designed for switched systems by solving the LMIIs under the switched signal which is larger than average dwell time, which differs from the commonly used parameter design method. Then, the residuals of switched systems are obtained and decoupled from unknown inputs based on UIOs. At last, the fault could be determined through evaluating the residuals. The main contribution of this paper is to give the designing condition of UIOs for switched system using LMI and average dwell time technique. The rest content of the paper is arranged as
following. In the section two, the model representation of switched system is described. The section three constructs UIOs, which is utilized for fault detection. Section four gives a simulation example to demonstrate effect of the new method. The conclusion is made in the section five.

**Notation:** $\mathbb{R}^n$ denotes $n$-dimension Euclidean space. $\|\|$ represents the Euclidean norm of vector. $\mathbb{N}$ is the set of positive integer. $\mathbb{N}^n$ represents the set of $n \times n$ dimension real matrices. T is symbol of transpose operation.

### 2. System Description

Let us consider discrete time linear switched system, which can be formulated as followed.

\[
\begin{align*}
    x(k+1) &= A_{\sigma(k)} x(k) + B_{\sigma(k)} u(k) + E_{\sigma(k)} d(k) + B_{\sigma(k)} f_{\sigma(k)}(k) \\
    y(k) &= C_{\sigma(k)} x(k)
\end{align*}
\]

(1)

where $x(k) \in \mathbb{R}^n$ denotes the system state, $y(k) \in \mathbb{R}^m$ is measured output, $u(k) \in \mathbb{R}^r$ is the control input, $d(k) \in \mathbb{R}^p$ is the unknown disturbance input, $f_{\sigma(k)}(k) \in \mathbb{R}^q$ represents actuator fault, which is additive. $\sigma(k) : [0, \infty) \rightarrow \mathbb{N} = \{1, 2, 3, \ldots, n\}$ represents the switched signal which is a piecewise constant function. $n > 0$ is the number of subsystem. $A_{\sigma(k)}, B_{\sigma(k)}, C_{\sigma(k)},$ and $E_{\sigma(k)}$ are the known constant matrices with proper dimensions. When $\sigma(k) = i (i \in \mathbb{N})$, it means that the $i$th subsystem $(A_i, B_i, C_i, E_i)$ is working. Then the formulation (1) could be converted as

\[
\begin{align*}
    x(k+1) &= A_i x(k) + B_i u(k) + E_i d(k) + B_i f_i(k) \\
    y(k) &= C_i x(k)
\end{align*}
\]

(2)

**Remark 1:** Because the location of fault is on the actuator, the matrix of fault is the same as the one of control input. When there is no fault, we let $f_i(k) = 0$, otherwise $f_i(k) \neq 0$.

### 3. Main Results

This section firstly designs unknown input observer for linear discrete switched system, assuming that the system has no fault occurring. The sufficient condition for the existence of UIOs and designing method of parameters are made. Then for the faulty switched system, residuals can be concluded by the UIOs, and be employed to achieve fault detecting with decoupling from unknown inputs.

#### 3.1. Design of Unknown Input Observers

Let $f_i(k) = 0$, then the model (2) is transformed as

\[
\begin{align*}
    x(k+1) &= A_i x(k) + B_i u(k) + E_i d(k) \\
    y(k) &= C_i x(k)
\end{align*}
\]

(3)

where $i \in \mathbb{N}$ denotes the $i$th subsystem. The structure of full rank UIOs can be formulated:

\[
\begin{align*}
    z(k+1) &= F_i z(k) + T_i B_i u(k) + K_i y(k) \\
    \hat{x}(k) &= z(k) + H_i y(k)
\end{align*}
\]

(4)

where $z(k)$ is the estimated value of system state $x(k)$, $z(k)$ is the state of full rank UIOs. $F_i, T_i, K_i,$ and $H_i$ are unknown matrices which need to be designed.

**Lemma 1** [14]: Equation (4) is UIOs of the switched system (3), if and only if

(i) $\text{rank}(C_i E_i) = \text{rank}(E_i)$
where

\[ A_{ii} = A_i - E_i ((C_i E_i)^T C_i E_i)^T C_i A_i \]  

According to the above, the formulation of UIOs is constructed for each subsystem. In the next part, the design of parameters for UIOs of switched system will be realized using the multiple Lyapunov function and average dwell time method. Before giving the main result, we need introduce some definitions and lemmas for demonstrating the theorem.

**Definition 1** [15]: There exist constants \( K > 0, 0 < \beta < 1 \) such that the solution \( x(k) \) of system (1) satisfies

\[
\|x(k)\| \leq K \beta^{(k-k_0)} \|x(k_0)\|, \quad \forall k \geq k_0,
\]

then the equilibrium \( x = 0 \) of system (1) is exponentially stable under switching signal \( \sigma(k) \).

**Definition 2** [15]: A positive constant \( \tau_{\alpha} \) is called the average dwell time for a switching signal \( \sigma(k) \) if

\[
N_\alpha(k, k_\tau) \leq N_\alpha + \frac{k - k_0}{\tau_{\alpha}}
\]

holds for all \( k_0 < k < k_\tau \), and some scalar \( N_\alpha > 0 \) that is called chatter bound.

where \( N_\alpha(k, k_\tau) \) means the number of mode switches of a given switching signal \( \sigma(k) \) over the interval \( (k, k_\tau) \).

**Remark 2**: For simplification, \( N_\alpha \) is always selected to equate to zero.

**Lemma 2** [16]: Consider the switched system (1), and let \( 0 < \alpha < 1, \mu > 1 \) be given constants. Suppose that there exist multiple Lyapunov function candidate \( V(x) = \{V_v(x) | \sigma \in \square\} \) satisfying the following properties:

\[
\begin{align*}
(a) & \quad V_{\alpha^{(i)}}(x(k + 1)) - V_{\alpha^{(i)}}(x(k)) \leq -\alpha V_{\alpha^{(i)}}(x(k)) \\
(b) & \quad V_{\alpha^{(i)}}(x(k)) \leq \mu V_{\alpha^{(j-1)}}(x(k))
\end{align*}
\]

Then the system (1) is globally exponentially stable for any switching signal with the average dwell time

\[
\tau_{\alpha} \geq \tau_{\alpha}^* = \text{ceil} \left[ -\frac{\ln \mu}{\ln (1 - \alpha)} \right]
\]

where function \( \text{ceil}(\nu) \) represents the nearest integer greater than or equal to \( \nu \) rounding real number \( \nu \).

Then, we will give our main result as following theorem.

**Theorem 1**: Given \( 0 < \alpha < 1, \mu > 1 \), if there exist systemic matrices \( P_i > 0 (\forall i \in \mathbb{Y}) \), and positive integer \( \tau_{\alpha}^* \) which denotes dwell time of switched signal make

\[
\begin{bmatrix}
-P_i & P_i F_i \\
F_i^T P_i & -(1 - \alpha) P_i
\end{bmatrix} \leq 0, \quad \forall i \in \mathbb{Y}
\]

\[ P_i \leq \mu, \quad (i \in \mathbb{Y}, j \in \mathbb{Y}, i \neq j) \]

\[
\tau_{\alpha} \geq \tau_{\alpha}^* = \text{ceil} \left[ -\frac{\ln \mu}{\ln (1 - \alpha)} \right]
\]

hold, and if
\[(H_iC_i - I)E_i = 0\]
\[T_i = I - H_iC_i\]
\[F_i = A_i - H_iC_iA_i - K_{n,i}C_i\]
\[K_{i,12} = F_iH_i\] (13)

hold, then equation (4) is UIOs of the switched system (3), and parameters of UIOs can be designed.

**Proof:** The error between state variable and the estimated state is defined as \(e(k) = x(k) -  \hat{x}(k)\). Through the original system (3) and UIOs (4), the dynamic equation can be derived as

\[
e(k + 1) = (A_i - H_iC_iA_i - K_{n,i}C_i)e(k) - [F_i - (A_i - H_iC_iA_i - K_{n,i}C_i)]z(k)
- [K_{i,12} - (A_i - H_iC_iA_i - K_{n,i}C_i)H_i]y(k)
- [T_i - (I - H_iC_i)]B_u(k) - (H_iC_i - I)E_d(k)
\] (14)

where

\[K_i = K_{i,1} + K_{i,2}\] (15)

In order to make the error decouple from known control input \(u(k)\), measured output \(y(k)\) and unknown input \(d(k)\), we should let

\[(H_iC_i - I)E_i = 0\]
\[T_i - (I - H_iC_i) = 0\]
\[F_i - (A_i - H_iC_iA_i - K_{n,i}C_i) = 0\]
\[K_{i,12} - (A_i - H_iC_iA_i - K_{n,i}C_i)H_i = 0\] (16)

Then the (13) can be illustrated. Furthermore, it can be concluded that

\[H_i = E_i[(C_iE_i)^T(C_iE_i)]^{-1}(C_iE_i)^T\] (17)
\[A_{i,i} = A_i - E_i[(C_iE_i)^T(C_iE_i)]^{-1}(C_iE_i)^T C_iA_i = A_i - H_iC_iA_i\] (18)
\[F_i = A_i - H_iC_iA_i - K_{n,i}C_i = A_{i,i} - K_{n,i}C_i\] (19)

Base on (6), the error equation can be reduced as

\[e(k + 1) = F_i e(k)\] (20)

It can be seen that the error system is also a switched system. We assume that the switched signal of error system is the same as the system (3), which is called synchronous switching. Next, we should derive the conditions which make the error system exponentially stable so as to design \(F_i\).

Construct the following multiple Lyapunov function for error system (20)

\[V_i(e(k)) = e^T(k)P_i e(k)\] (21)

where \(i = 1, 2, \ldots, n\). Assume \(P_i = P_i^T > 0\), it is obvious that \(V_i(e(k)) > 0\) holds. Then we will calculate \(\Delta V\). Based on the Lemma2, \(\Delta V\) need satisfy two conditions:

\[V_i(e(k + 1)) - V_i(e(k)) = e^T(k)F_i^T P_i F_i e(k) - e^T(k)P_i e(k) \leq - \alpha e^T(k)P_i e(k)\]
\[\Rightarrow e^T(k)(F_i^T P_i F_i - (1 - \alpha)P_i) e(k) \leq 0\]
\[V_i(x(k)) \leq \mu V_i(x(k)) \Rightarrow P_i \leq \mu P_i\] (22)
where \( i \in \mathbb{V}, j \in \mathbb{V}, i \neq j \), the \( k_i, k_{i+1} \) denote the time when the \( i \)th subsystem is activated, and \( k_j \) represents the switched time when the \( j \)th subsystem switches to the \( j \)th subsystem.

Therefore, if the following inequalities
\[
F_i^T P_i F_i - (1 - \alpha)P_i \leq 0
\]
\[
P_i \leq \mu P_j
\]
are satisfied, and dwell time of switched signal
\[
\tau_a \geq \tau_a^* = \text{ceil} \left[ \frac{-\ln \mu}{\ln(1 - \alpha)} \right]
\]
holds, then \( \Delta V < 0 \). On the basis of Lemma 2, the error system (20) is exponentially stable. And according to Schur complement lemma, (24) equates to
\[
\begin{bmatrix}
-P_i & P_i F_i \\
F_i^T P_i & -(1 - \alpha)P_i
\end{bmatrix} < 0
\]
Namely, the proof is done.

Substitute \( F_i = A_i - K_i C_i \) into the above matrix inequality, we get
\[
\begin{bmatrix}
-P_i & P_i (A_i - K_i C_i) \\
(A_i - K_i C_i)^T P_i & -(1 - \alpha)P_i
\end{bmatrix} < 0
\]
There are nonlinear terms, which need to be transformed to the form of linear matrix inequalities. Define \( W_\alpha = P_i K_i \), (27) is converted as
\[
\begin{bmatrix}
-P_i & P_i A_i - W_\alpha C_i \\
A_i^T P_i - W_\alpha^T & -(1 - \alpha)P_i
\end{bmatrix} < 0
\]
In the (29), \( P_i \) and \( W_\alpha \) can be solved by LMI toolbox of Matlab directly. Since \( K_i = P_i^{-1} W_\alpha \), we can obtain the value of \( K_i \), then \( F_i, K_i \), and \( K_i = K_i + K_i \) also are determined naturally. So far, the proof is finished.

### 3.2. Fault Detection Based on UIOs

In this section, we will consider switched system (2) which owns actuator fault, and construct residual generator by UIOs. First of all, we define residual signal as
\[
r_i(k) = y(k) - C_i \hat{x}(k)
\]
where \( i = 1, 2, \ldots, n \). Taking fault into account, the error equation is updated as
\[
e(k + 1) = (A_i - H_i C_i A_j - K_j C_j) e(k) - [F_i - (A_i - H_i C_i A_j - K_j C_j) z(k)]
\]
\[
- [K_{ii} - (A_j - H_j C_j A_j - K_j C_j) H_j] y(k)
\]
\[
- [T_j - (I - H_j C_j)] B_j u(k) - (H_j C_j - I) E_j d(k)
\]
\[
+ (I - H_j C_j) B_j f_j(k)
\]
According to Theorem 1, residual generator can be formulated and decouple from known control input $u(k)$, measured output $y(k)$ and unknown input $d(k)$.

$$e(k+1) = F_e(k) + T_1 B_f f_1(k)$$
$$r_f(k) = C_f e(k)$$  \hspace{1cm} (32)

In addition, for detecting actuator fault, $T B_i 
eq 0$ must hold at the same time. When the error system (32) is stable, the residual only is affected by fault. The residual of whole switched system is composed of one of each subsystem. That is

$$\begin{cases}
    r(k) = r_1(k_1) U r_2(k_2) U L U r_n(k_n) \\
    k = k_1 U k_2 U L U k_n
\end{cases} \hspace{1cm} (33)$$

where $k_1, k_2, \ldots, k_n$ are the total time of running subsystem separately.

The next step is residual evaluation. Firstly, calculate Euclidean norm of residual

$$J_f(r) = \|r(k)\|_2 = \sqrt{\sum_{i=1}^L r(k)^T r(k)}$$  \hspace{1cm} (34)

where $L$ is the evaluation time window.

Then the threshold is specified as

$$J_{th} = (\sup_{r(k)} \|r(k)\|_2) \bigg|_{\|r(k)\|_2 = 0} \hspace{1cm} (35)$$

When $J_f(r) > J_{th}$, there is a fault. When $J_f(r) \le J_{th}$, there is no fault.

4. Simulation Example

In this section, consider a discrete linear switched system which includes two subsystems. The model of each subsystem is the same as equation (2), and the parameter matrices are given as

$$A_1 = \begin{bmatrix}
-1 & 1.4 & 0 \\
-1 & 0 & 0 \\
0 & -1 & -1
\end{bmatrix}, A_2 = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -0.9 & -1
\end{bmatrix} \hspace{1cm} (36)$$

$$B_1 = \begin{bmatrix}
0.8 & 0.9 \\
1 & 0.8
\end{bmatrix}^T, B_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix} \hspace{1cm} (37)$$

$$C_1 = \begin{bmatrix}
1.2 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}, C_2 = \begin{bmatrix}
0.9 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \hspace{1cm} (38)$$

$$E_1 = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}^T, E_2 = \begin{bmatrix}
-1.5 & 0 & 0 \\
-1 & -1 & -1
\end{bmatrix} \hspace{1cm} (39)$$

Let $\alpha = 0.2, \mu = 1.2$, then average dwell time can be calculated as

$$\tau'_a = \text{ceil} \left[ -\frac{\ln \mu}{\ln(1-\alpha)} \right] = \text{ceil} \left[ -\frac{\ln 1.2}{\ln 0.8} \right] = 1$$  \hspace{1cm} (40)

$H_i$ and $A_{ii}$ are determined as

$$H_1 = \begin{bmatrix}
0.4918 & 0.4098 \\
0 & 0 \\
0 & 0
\end{bmatrix}, H_2 = \begin{bmatrix}
1.1111 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \hspace{1cm} (41)$$
Verify whether the system fulfills the condition of Lemma 1. Calculate the rank of $C_E$ and $E_i$

\[
\begin{align*}
\text{rank}(C_E) &= \text{rank}(E_i) = 1 \\
\text{rank}(C_2E_2) &= \text{rank}(E_2) = 1
\end{align*}
\]

Then $(C_1A_1)$ and $(C_2A_2)$ are both observable. $k \in [1,300]$ is the total running time of whole switched system. The switched signal is given in Figure 1. Let dwell time $
\tau_d = 2 > \tau_d = 1$

![Figure 1. The Switched Signal](image)

The known input $u(k)$ equates to 2 throughout running time. The unknown input signal $d(k) = 0.1 \times \sin(0.1 \times k)$, which is shown in Figure 2.

![Figure 2. Unknown Input Signal](image)

And the fault signal is described as

\[
f_f(k) = \begin{cases} 
0, & k \in [1,150] \\
0.01, & k \in [151,170] \\
0, & k \in [171,300]
\end{cases}
\]

Applying Theorem 1, the rest parameters of UIOs are calculated as below:

\[
F_i = \begin{bmatrix} 0 & 0.9016 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -0.9 & 0 \end{bmatrix}
\]
From the above, we can get residual signal based on UIOs. Compute the residual evaluating function and the threshold, that is

\[
J_{i_0} = \left( \sup_{r(k) \neq 0} \| r(k) \|_{T_{i_0}} \right) = \sqrt{\sum_{k=1}^{500} r^T(k) r(k)} = 0.00001
\]

(48)

\[
J_{L}(r) = \left( \| r(k) \|_{T_{L}} \right)_{r(k) \neq 0} = \sqrt{\sum_{k=1}^{152} r^T(k) r(k)} = 0.008
\]

(49)

For \( J_{L}(r) > J_{i_0} \), the fault can be detected. It is also verified in Figure 3, which represents the evolution of residual evaluating function. The red solid curve denotes the evolution of residual evaluating function, and the black dash line denotes the threshold.

![Figure 3. Evolution of Residual Evaluating Function](image)

5. Conclusions

In the paper, the fault detection problem of linear switched systems is achieved through constructing residual generator using UIOs and average dwell time approach. The design conditions of UIOs are given in the form of LMIs, which are convenient to solve. At last, a simulation case is used for verifying the effective of method.

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References

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