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Active Adaptive Sliding Mode Control of Fractional-Order Hyperchaotic Systems with Uncertainties and External Disturbances

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Abstract

This work presents an active adaptive sliding mode controller for using synchronization of fractional-order hyperchaotic systems with uncertainties and external disturbances. Based on Lyapunov stability theorems a fractional sliding surface is designed and an active adaptive sliding mode controller is proposed. In this method, the bound of the uncertainties and external disturbances are estimated through the adaptive updating law. The synchronization scheme is global and theoretically rigorous. Simulation studies have shown the proposed controllers can get good control effects.

Keywords: *fractional-order hyperchaotic system, synchronization, anti-synchronization, active adaptive sliding model, uncertainties and external disturbances*

1. Introduction

Recently, fractional-order dynamics have attracted considerable attention due to the fact that fractional differential equations can describe many physical systems more accurately [1]. Synchronization of fractional-order chaotic systems starts to receive increasing attention due to its potential applications in secure communication and control processing. With contribution from the foremost researchers in synchronization of fractional-order chaotic systems there has been successful achievements such as PC control [2], sliding mode control [3], adaptive control [4,5], nonlinear state observer method [6] nonlinear feedback control [7, 8] *etc*. However, most of current results have concentrated on low-dimensional chaotic system. The research of the hyperchaotic system usually defined as a chaotic system with more than one positive Lyapunov exponent is less by far. Hyperchaotic system has more complex dynamical behaviors than chaotic system and is more suitable in secure communications. So study on hyperchaotic synchronization is very important and is a more challenging research [9, 10].

So far only a few synchronization methods such as feedback controller and nonlinear controller have been proposed [11-13]. Most of the above synchronization methods seldom deal with uncertainties and external disturbances. The existence of uncertainties and external disturbances is often the cause of poor performance, undesirable system transient response, and instability. Therefore, in this paper, we present a new fractional order controller to synchronize the hyperchaotic systems with uncertainties and external disturbances. The adaptive updating law is designed to estate the bound of the uncertainties and external disturbances under the combination of active sliding mode control and adaptive control. Synchronization and anti-synchronization are converted via changed value of the parameter in the control function. The stability of error dynamics are demonstrated based on the Lyapunov stability theory. Numerical simulation of

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hyperchaotic system illustrates the effectiveness and superiority of the proposed control method.

2. Preliminaries and Definitions

Definitions for fractional derivatives can be found in many textbook on fractional calculus [14-16]. The Caputo derivative is often used [11]:

$$D_t^{\alpha} x(t) = J^{m-\alpha} x^{(m)}(t) \quad \text{with } \alpha > 0 , \tag{1}$$

where $m = [\alpha]$, *i.e.*, m is the first integer which is not less than α ; $x^{(m)}$ is the general m -order derivative, and J^{β} ($\beta > 0$) is the β -order Riemann-Liouville integral operator expressed as follows:

$$J^{\beta} y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} y(\tau) d\tau \tag{2}$$

where Γ stands for the Gamma function.

The general properties of the fractional-order calculus are recalled below [17, 18].

Property 1.

$$D_t^{\alpha} x(t) = D_t^{1-\alpha} x(t) D_t^{\alpha} x(t) \tag{3}$$

where $0 < \alpha < 1$, and $D = \frac{d}{dt}$.

Property 2.

$$D_t^{\alpha} (\lambda x(t) + \mu y(t)) = \lambda D_t^{\alpha} x(t) + \mu D_t^{\alpha} y(t) \tag{4}$$

where λ, μ are real constants .

3. Active Adaptive Sliding Mode Controller Design and Analysis

Consider the following drive-response systems described by (5) and (6)

$$D_t^{\alpha} x = (A_1 + \Delta A_1) x + f_1(x) + \delta_1(t) \tag{5}$$

and

$$D_t^{\alpha} y = (A_2 + \Delta A_2) y + f_2(y) + \delta_2(t) + u(t) \tag{6}$$

where $0 < \alpha < 1$, $x, y \in R^n$ are the n-dimensional state vectors of the system. $A_1, A_2 \in R^{n \times n}$ are the linear parts of the system dynamics and $f_1(x), f_2(y)$ are the nonlinear parts of the system. $\Delta A_1, \Delta A_2 \in R^{n \times n}$ are the matrixes of uncertainties and $\delta_1(t), \delta_2(t) \in R^{n \times n}$ are the vectors denoting external disturbances. In (6), $u(t) \in R^n$ is the output of the controller.

Remark: If $A_1 = A_2$ and $f_1(x) = f_2(x)$, drive system and response system are two identical chaotic systems that also can be called hyperchaotic systems. Otherwise they represent two different hyperchaotic chaotic systems.

Suppose

$$e = y + \beta x \tag{7}$$

where $\beta = [\beta_1, \beta_2, \cdots \beta_{n-1}, \beta_n]^T = [-1, -1, \cdots -1, -1]^T$, the synchronization type is complete synchronization and if $\beta = [\beta_1, \beta_2, \cdots \beta_{n-1}, \beta_n]^T = [1,\ 1,\ \cdots 1,\ 1]^T$, the synchronization type is anti-synchronization.

The error system can be rewritten as follows:

$$\begin{aligned} D_t^{\alpha} e &= (A_2+\Delta A_2)y + f_2(y) + \delta_2(t) + u(t) + \beta((A_1+\Delta A_1)x \\ &\quad + f_1(x) + \delta_1(t)) + u(t) \\ &= (A_2+\Delta A_2 + \Delta A_1)e + \delta_2(t) + \beta\delta_1(t) + f_2(y) + \beta f_1(x) \\ &\quad + \beta(A_1 - (A_2+\Delta A_2))x - \Delta A_1 y + u(t) \end{aligned} \tag{8}$$

For simplicity, the following assumption is made:

$$F(x, y) = f_2(y) + \beta f_1(x) + \beta(A_1 - (A_2+\Delta A_2))x - \Delta A_1 y + u(t) \tag{9}$$

So (8) is expressed as:

$$D_t^{\alpha} e = (A_2+\Delta A_2 + \Delta A_1)e + \delta_2(t) + \beta\delta_1(t) + F(x, y) + u(t) \tag{10}$$

3.1. Controller Design

In order to achieve the synchronization, an appropriate control added into the response system is designed to make error zero. That means $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y + \beta x\| = 0$.

Before proceeding with the main result of this paper, the following assumption, which specify the class of chaos with uncertainty is made.

Assumption 1. $\|\Delta A_2 + \beta\Delta A_1\| < \psi$ and $\|\delta_2 + \beta\delta_1\| < \eta$, where ψ, η are unknown positive constants.

The overall active adaptive sliding mode control scheme is shown in (11).

According to the active control design procedure, the output of controller is used to eliminate the nonlinear part of the error dynamics. Therefore the $u(t)$ is considered as

$$u(t) = G(t) - F(x, y) \tag{11}$$

Hence the error system can be rewritten as:

$$D_t^{\alpha} e = (A_2+\Delta A_2 + \Delta A_1)e + \delta_2(t) + \beta\delta_1(t) + G(t) \tag{12}$$

There are many possible methods for the control input *G(t)*. Without loss of generality, we choose the sliding mode control law as follows:

$$G(t) = Kv(t) \tag{13}$$

where $K = [k_1, k_2, \cdots, k_{n-1}, k_n]^T$ is a constant gain vector and $v(t)$ satisfies

$$v(t) = \begin{cases} v^+(t) & s \ge 0 \\ v^-(t) & s < 0 \end{cases} \tag{14}$$

where $s = s(e)$ is a switching surface.

The sliding mode control method involves two major stages:

Step1. Choose a suitable sliding surface.

Step2. Design the sliding mode controller.

The sliding surface in the error state space is defined as:

$$s = D_t^{\alpha-1} Ce \tag{15}$$

where $C = [C_1, C_2, \cdots, C_{n-1}, C_n]^T$ is a constant vector.

When in sliding surface, the controlled system should satisfy the following conditions:

$$s = D_t^{\alpha-1} Ce = 0 \text{ and } \dot{s} = Ds = CD_t^{\alpha} e = 0 \tag{16}$$

Generally the sliding mode control method applies the constant plus proportional rate reaching law. The reaching law is expressed as:

$$\dot{s} = -\varepsilon \operatorname{sgn}(s) - rs \tag{17}$$

where ε , r are positive real numbers.

Hence, the $v(t)$ can be expressed as follow:

$$v(t) = -(CK)^{-1}[CA_2e + C\Delta A_2 y + \beta C\Delta A_1 x + C\delta_2(t) + \beta C\delta_1(t) + rs + \varepsilon \cdot \operatorname{sgn}(s)] \tag{18}$$

where the existence of $(CK)^{-1}$ is a necessary condition.

There exist system uncertainties and external disturbances in (18). In this regard, we propose the following control law as follow:

$$v(t) = -(CK)^{-1}[CA_2e + \|C\|\hat{\psi} \operatorname{sgn}(s) + \|C\|\hat{\eta} \operatorname{sgn}(s) + rs + \varepsilon \cdot \operatorname{sgn}(s)] \tag{19}$$

To tackle the bounds of the error system uncertainties and external disturbances, the suitable adaptive laws are defined as follow:

$$\dot{\hat{\psi}} = -\|Cs\|, \dot{\hat{\eta}} = -\|Cs\| \tag{20}$$

where $\hat{\psi}, \hat{\eta}$ are estimations for ψ, η of assumption 1.

3.2. Stability Analysis

To prove that the error dynamics (8) is asymptotically stable, we choose the Lyapunov function defined by the equation

$$V = \frac{1}{2} s^2 + \frac{1}{2}\tilde{\psi}^2 + \frac{1}{2}\tilde{\eta}^2 \tag{21}$$

Obviously, V is a positive definite function on R^n, where $\tilde{\psi} = \hat{\psi} + \psi, \tilde{\eta} = \hat{\eta} + \eta$.

Taking derivative of the Lyapunov function candidate with respect to time, one has

$$
\begin{aligned}
\dot{V} &= s\dot{s} + \tilde{\psi}\dot{\tilde{\psi}} + \tilde{\eta}\dot{\tilde{\eta}} \\
&= sC(A_2 e - K(CK)^{-1}(CA_2 e + \|C\|\hat{\psi}\,\mathrm{sgn}(s) \\
&+ \|C\|\hat{\eta}\,\mathrm{sgn}(s) + \varepsilon\,\mathrm{sgn}(s) + rs) + \Delta A_2 y \\
&+ \beta\Delta A_1 x + \delta_2(t) + \beta\delta_1(t)) + \tilde{\psi}\dot{\tilde{\psi}} + \tilde{\eta}\dot{\tilde{\eta}} \\
&= -\varepsilon s\,\mathrm{sgn}(s) - rs^2 - \|C\|\hat{\psi} s\,\mathrm{sgn}(s) \\
&- \|C\|\hat{\eta} s\,\mathrm{sgn}(s) + sC(\Delta A_2 y + \beta\Delta A_1 x) \\
&+ sC(\delta_2(t) + \beta\delta_1(t)) + \tilde{\psi}\dot{\tilde{\psi}} + \tilde{\eta}\dot{\tilde{\eta}} \\
&\le -\varepsilon s\,\mathrm{sgn}(s) - rs^2 - \|C\|\hat{\psi} s\,\mathrm{sgn}(s) \\
&- \|C\|\hat{\eta} s\,\mathrm{sgn}(s) + \|Cs\|\|\Delta A_2 y + \beta\Delta A_1 x\| \\
&+ \|Cs\|\|\delta_2(t) + \beta\delta_1(t)\| + \tilde{\psi}\dot{\tilde{\psi}} + \tilde{\eta}\dot{\tilde{\eta}} \\
&\le -\varepsilon s\,\mathrm{sgn}(s) - rs^2 - \|C\|\hat{\psi} s\,\mathrm{sgn}(s) \\
&- \|C\|\hat{\eta} s\,\mathrm{sgn}(s) + \|Cs\|\psi + \|Cs\|\eta + \tilde{\psi}\dot{\tilde{\psi}} + \tilde{\eta}\dot{\tilde{\eta}} \\
&= -\varepsilon s\,\mathrm{sgn}(s) - rs^2 - \|Cs\,\mathrm{sgn}(s)\|\hat{\psi} - \|Cs\|\hat{\eta} \\
&- \|Cs\|\hat{\psi} - \|Cs\|\hat{\eta} \le 0
\end{aligned} \tag{22}
$$

According to Barbalat's lemma [19], it can be obtained that $s, \dot{s} \in L_\infty$. Furthermore, according to Lyapunov stability theory, $\lim_{t\to\infty} s = 0$. Therefore the error e asymptotically converges to zero and that the two systems (5) and (6) are globally asymptotically synchronized.

Remark: This section uses the adaptive updating law to estate the bound of the uncertainties and external disturbances, which has great significance in controller implemention.

4. Numerical Simulation

In this section, two numerical simulation results are discussed to validate the effectiveness of the two proposed synchronization scheme.

The fractional order hyperchaotic Chen system is written as

$$
\begin{cases}
D_t^\alpha x_1 = 35(x_2 - x_1) + x_4 \\
D_t^\alpha x_2 = 7x_1 - x_1 x_3 + 12x_2 \\
D_t^\alpha x_3 = x_1 x_2 - 3x_3 \\
D_t^\alpha x_4 = x_2 x_3 + 0.5x_4
\end{cases} \tag{23}
$$

where $0 < \alpha < 1$ is the fractional order time derivatives. The lowest value of α for which the system remains chaotic is 0.96.

The fractional order hyperchaotic Lorenz system is written as

$$
\begin{cases}
D_t^\alpha x_1 = 10(x_2 - x_1) + x_4 \\
D_t^\alpha x_2 = -x_1 x_3 + 28x_1 - x_2 \\
D_t^\alpha x_3 = x_1 x_2 - (8/3)x_3 \\
D_t^\alpha x_4 = -x_1 x_2 + 1.3x_4
\end{cases} \tag{24}
$$

where $0 < \alpha < 1$ is the fractional order time derivatives. The lowest value of α for which the system remains chaotic is 0.96.

Example 1.Complete synchronization between two identical fractionl order hyperchaotic Chen systems.

When $\beta = [\beta_1, \beta_2, \cdots \beta_{n-1}, \beta_n]^T = [-1, -1, \cdots -1, -1]^T$, the synchronization type is complete synchronization. Now the drive system and the response system are defined with uncertainties and external disturbances.

$$\begin{cases} D_t^\alpha x_1 = 35(x_2 - x_1) + x_4 + 3x_1 + x_2 + x_4 - 0.5\cos(50t) \\ D_t^\alpha x_2 = 7x_1 - x_1x_3 + 12x_2 - 2x_2 + 0.5\sin(50t) \\ D_t^\alpha x_3 = x_1x_2 - 3x_3 + x_1 + \sin(50t) \\ D_t^\alpha x_4 = x_2x_3 + 0.5x_4 + x_3 - \sin(50t) \end{cases} \tag{25}$$

And

$$\begin{cases} D_t^\alpha y_1 = 35(y_2 - y_1) + 2y_4 + 2y_1 + y_2 - 0.5\cos(50t) + u_1(t) \\ D_t^\alpha y_2 = 7y_1 - x_2y_3 + 12y_2 - 2y_2 + 0.3\sin(50t) + u_2(t) \\ D_t^\alpha y_3 = y_1y_2 - 3y_3 + 2y_1 - 0.2\sin(50t) + u_3(t) \\ D_t^\alpha y_4 = y_2y_3 + 0.5y_4 + y_3 - 0.2\sin(50t) + u_4(t) \end{cases} \tag{26}$$

The corresponding numerical results are shown in Figure 1-2, where the initial values are set as $x(0) = (1, 5.8, -1, 2)$, $y(0) = (2, 2, -1, 2)$, $\psi(0) = 28$, $\eta(0) = -31, C = (1,\ 2,\ 1,\ \text{-}1)$ and $K = (1,\ 1,\ 0,\ 1)^T$. Figure 1 shows the synchronization error between systems (25) and (26), which show that the synchronization can be achieved. Figure 2 represents the estimates of $\hat{\eta}, \hat{\psi}$, which shows that the bounds of uncertainties and external disturbances are estimated.

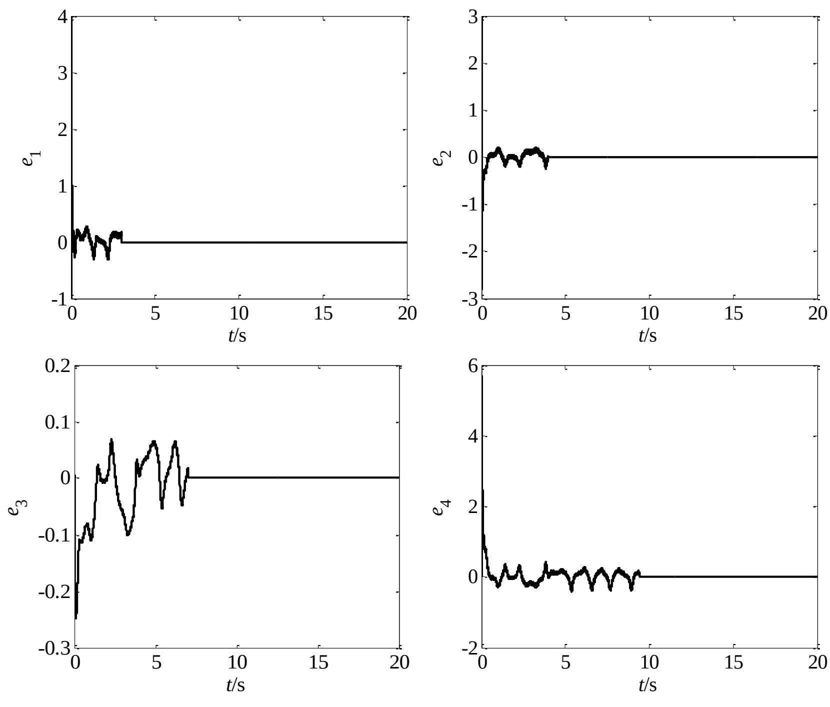

Figure 1. Synchronization Error *e* of the Chen Systems

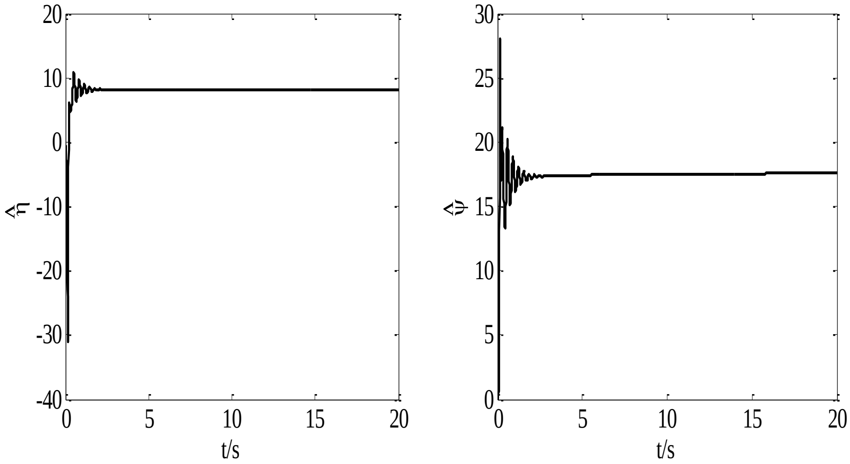

Figure 2. Estimated Parameter of η and Ψ

Example 2. Anti-synchronization between fractionl order hyperchaotic Lorenz systems and Chen system.

When $\beta = [\beta_1, \beta_2, \cdots \beta_{n-1}, \beta_n]^T = [1,1,\cdots 1,1]^T$, the synchronization type is anti-synchronization. Now the drive system and the response system are defined with uncertainties and external disturbances.

$$\begin{cases} D_t^\alpha x_1 = 35(x_2 - x_1) + x_4 + 3x_1 + x_2 + x_4 - 0.5\cos(50t) \\ D_t^\alpha x_2 = 7x_1 - x_1x_3 + 12x_2 - 2x_2 + 0.5\sin(50t) \\ D_t^\alpha x_3 = x_1x_2 - 3x_3 + x_1 + \sin(50t) \\ D_t^\alpha x_4 = x_2x_3 + 0.5x_4 + x_3 - \sin(50t) \end{cases} \tag{27}$$

The slave system can be rewritten as the following

$$\begin{cases} D_t^\alpha y_1 = 10(y_2 - y_1) + y_4 + 3y_1 + y_2 - 0.5\cos(50t) + u_1(t) \\ D_t^\alpha y_2 = -y_1y_3 + 28y_1 - y_2 - 2y_2 + 0.5\sin(50t) + u_2(t) \\ D_t^\alpha y_3 = y_1y_2 - 28/3y_3 + y_1 + \sin(50t) + u_3(t) \\ D_t^\alpha y_4 = -y_1y_3 + 1.3y_4 + y_3 - \sin(50t) + u_4(t) \end{cases} \tag{28}$$

The corresponding numerical results are shown in Figure 3-4, where the initial values are set as $x(0) = (-1,3,-0.8,-2)$, $y(0) = (-1.1,2.5,-0.8,1.6)$, $\psi(0) = -26$, $\eta(0) = -12$, $C = (0,\ 2,\ 1,\ -1)$ and $K = (1,\ 1,\ 0,\ 1)^T$. Figure 3 shows the synchronization error between systems (27) and (28), which shows that the synchronization can be achieved. Figure 4 represents the estimates of $\hat{\eta}, \hat{\psi}$, which shows that the bounds of uncertainties and external disturbances are estimated.

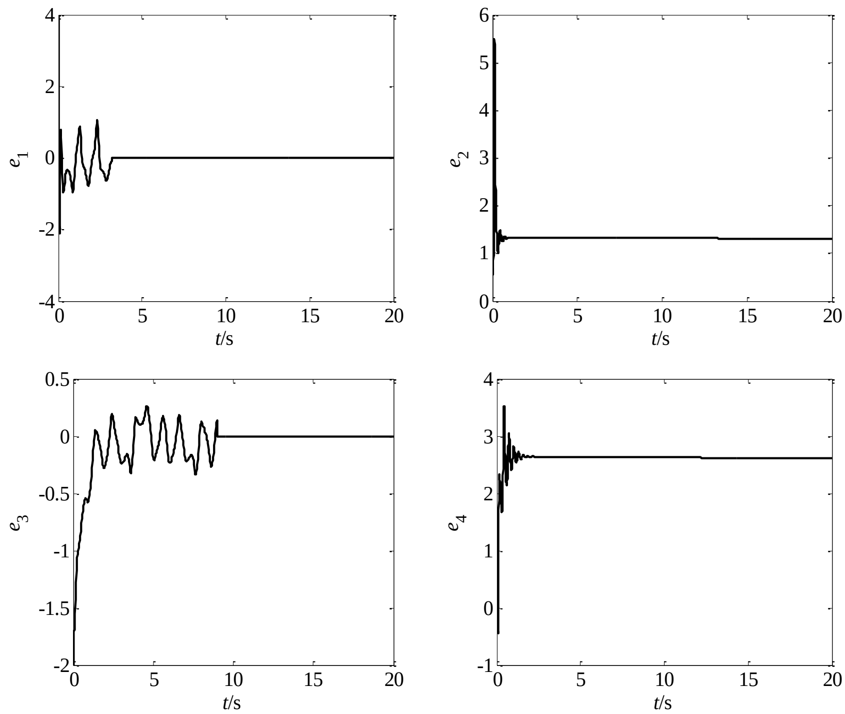

Figure 3. Synchronization Error e of the Chen Systems and Lorenz Systems

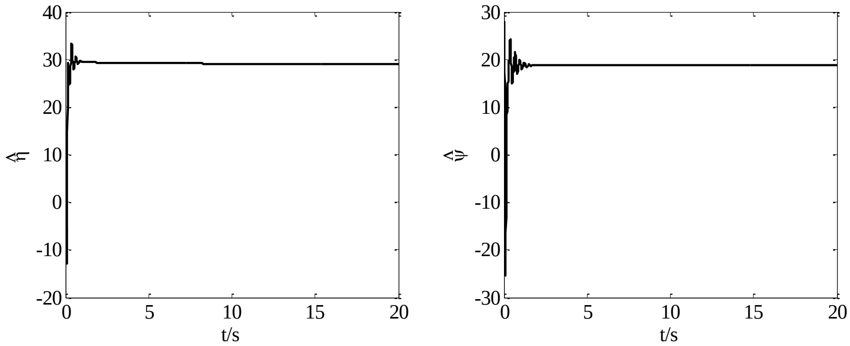

Figure 4. Estimated Parameter of η and Ψ

5. Conclusions

In this paper, based on Lyapunov stability theorems an active adaptive sliding mode controller has been proposed to synchronize fractional-order hyperchaotic systems with uncertainties and external disturbances. The proposed controller is simple and theoretically rigorous. Simulation results have illustrated the effectiveness of the theorem and the proposed method.

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