

# An EOQ Model for Perishable Items with Freshness-dependent Demand and Partial Backlogging

Xiaoming Yan

*School of Computer, Dongguan University of Technology, Guangdong, P. R. China  
yanxiaoming325@126.com*

## **Abstract**

*In this paper, we present an inventory model for perishable items with freshness-dependent demand rate and partial backlogging. Because the customers prefer to purchasing fresh items for given price, the demand rate is assumed to be decreasing in the age of the items. The existence and uniqueness of the optimal solution of the profit per unit time are examined although the profit function is not jointly concave. The results are illustrated through numerical example and sensitivity analysis is reported.*

**Keywords:** *Inventory, Perishable Items, EOQ, Backlogging*

## **1. Introduction**

In the classical economic order quantity (EOQ) model, it is often assumed that the demand rate is constant over time and the backlogging rate is often the fraction of the demand rate. However, as a physical phenomenon, the demand rate for a perishable item is often dependent on the freshness of the item during the sales horizon because the customers always prefer to purchasing fresher items given the same price of the item, and moreover, the waiting customers sometimes give up the waiting decisions during the interval of shortage. In this study, we incorporate the effect of the freshness (age) of a perishable item on the demand rate and assume that the actual demand rate is increasing (decreasing) in the freshness (age) of the item. The purpose of this paper is to analyze the optimal ordering policy for an EOQ model with freshness-dependent demand and partial backlogging. We not only obtain the sufficient and necessary condition which the optimal solution of the profit per unit time satisfies but also show that the optimal solution is unique on the whole decision domain.

In the last several decades, the EOQ model for perishable items has received the attention of many researchers. Padmanabhan and Vrat [1] presented EOQ models for perishable items with stock dependent selling rate. In their models, the selling rate is assumed to be a function of current inventory level and the rate of deterioration is taken to be constant. Based on Padmanabhan and Vrat's work, Chung, et al., [2] also analyzed EOQ models for deteriorating items with stock dependent selling rate. They developed the necessary and sufficient conditions of the existence and uniqueness of the optimal solutions of the profit per unit time without backlogging and with complete backlogging. Dye and Ouyang [3] extended Padmanabhan and Vrat [1]'s model by proposing a time-proportional backlogging rate. In their model, the willingness of a customer to wait for backlogging during a shortage period is decline with the length of the waiting time. Chen [4], Skouri and Papachristos [5], Wang [6], Papachristos and Skouri [7, 8] considered EOQ models for perishable items where the backlogging rate is a time-dependent function. Lee and Wu [9, 10], Wu, et al., [11] considered inventory models in which inventory is depleted not only by the deterministic demand, but also by deterioration. Min and Zhou [12] developed a deterministic inventory

model for perishable items with stock-dependent selling rate. Lin, et al., [13] explored the inventory replenishment policies for the cases with time-varying demand, linearly increasing deterioration rate, partial back-ordering, constant service level and equal replenishment intervals over a fixed planning horizon. Giri, et al., [14] studied an inventory model for deteriorating items with stock-dependent demand rate. Giri and Chaudhuri [15] dealt with an extended EOQ-type inventory model for a perishable product where the demand rate is a function of the on-hand inventory. Goyal and Giri [16] considered the production-inventory problem in which the demand, production and deterioration rates of a product are assumed to vary with time and the shortages of a cycle are allowed to be backlogged partially. Wu [17] studied EOQ inventory model for items with Weibull distribution deterioration, time-varying demand and partial backlogging. Shah and Shukla [18], Skouri, et al., [19] presented deteriorating inventory models for waiting time partial backlogging. Moreover, there have been many EOQ models considering the inflation or the permissible delay in payments. For example, Chang and Dye [20], Chang, et al., [21] considered a finite time horizon inventory model with deterioration and time-value of money under the conditions of permissible delay in payments. Hou and Lin [22] applied a discounted cash flow (DCF) approach for the analysis of a replenishment problem over a finite planning horizon. Basu and Sinha [23] presented a general inventory model with due consideration to the factors of time dependent partial backlogging and time dependent deterioration. Moon, et al., [24] developed models for deteriorating items with time varying demand pattern over a finite planning horizon, taking into account the effects of inflation and time value of money. Shah [25] derived an inventory model by assuming constant rate of deterioration of units in an inventory, time value of money under the conditions of permissible delay in payments. In this paper, we assume that the selling rate for perishable items is increasing in the freshness of the items since the customers like fresher items and in turn the items decay with time. The present work aims to explore the behavior of the profit function per unit time where selling rate depends on the age of the item and the backlogging will give up waiting during the interval of shortage.

There are many other works about perishable inventory. Nahmias [26, 27] and Pierskalla [28, 29] studied perishable inventory on determining the optimal ordering policies for fixed life items. Ghare and Schrader [30] analyzed the effect of constant rate of deterioration on inventory and generalized the Wilson [31]'s EOQ model without shortages. Dave and Patel [32] considered an inventory model with time-varying demand. They analyzed a linear increasing demand rate over a finite horizon and a constant deterioration rate. Sachan [33] extended Dave and Patel's model for shortages. Goswami and Chaudhuri [34], Hariga [35], Wu [36] developed EOQ model that focused on deteriorating items with time-varying demand and shortage over a finite horizon. Chang and Dye [37] presented an EOQ model for deteriorating items with time-varying demand and partial backlogging over a finite horizon. They focused on the effect of the backlogging rate on the economic order quantity decision. Dye [38] presented an model about joint pricing and ordering policy for a deteriorating inventory with partial backlogging. In their model, demand rate is not related to time if the price is given. Many EOQ models on deteriorating items can be found in Goyal and Giri [39].

The rest of this paper is organized as follows. In Section 2, the assumptions and notation related to this paper are presented. In Section 3, we present the mathematical model and then analyze the retailer's objective. In Section 4, the optimal solution of the profit function per unit time is analyzed. In Section 5, the model is illustrated with numerical example and in the last section we provide several conclusions and future research directions.

## 2. Assumptions and Notation

1. A single perishable item is replenished, stored and sold;
2. Replenishment rate is infinite and transportation time of items replenished is ignored;
3.  $t_1$  is the time up to which inventory is positive in a cycle,  $t_1 + t_2$  is the cycle time.
4.  $c$ , the marginal replenishment cost per unit and  $p > c$ , the selling price per unit, are known and constant;
5.  $h$ , the holding cost per unit per unit time and  $K$ , the fixed ordering cost, are known and constant;
6.  $c_1$ , the shortage cost per unit per unit time and  $c_2$ , the opportunity cost of lost sales per unit, are known and constant;
7.  $\theta$  is the constant deterioration (wastage) rate;
8.  $I(t)$  is the inventory level on hand at time  $t \in [0, t_1)$  and  $W(t)$  is the backlogging at time  $t \in [t_1, t_1 + t_2]$ ;
9.  $m(t)$  is the freshness of the item at time  $t$  and we assume that  $dm(t) = -\mu m(t)dt$  with  $m(0) = 1$ , where  $t \in [0, t_1)$  and  $\mu$  is a constant factor and known. Then the decreasing slope of the freshness of the item is directly proportional to the current freshness of the item. By calculating, we derive  $m(t) = e^{-\mu t}$  for  $t \in [0, t_1)$ . Since the consumers prefer to purchasing fresher items given the same price, the actual demand rate  $D(t)$  at time  $t \in [0, t_1)$  is assumed to be dependent on the freshness of the item and

$$D(t) = \begin{cases} \alpha e^{-\mu t} & 0 \leq t < t_1, \\ \alpha & t_1 \leq t \leq t_1 + t_2 \end{cases} \quad (1)$$

where  $\alpha$ ,  $\beta$  are constant and known.

10. During the shortage period  $[t_1, t_1 + t_2]$ , the backlogging,  $W(t)$ , satisfies the following expression:

$$\frac{dW(t)}{dt} = \alpha - \gamma W(t) \quad (2)$$

with  $W(t_1) = 0$ , where  $\gamma$  is the fraction of waiting customers giving up waiting decisions. Such form of partial backlogging has been used by many researchers, such as Sunil and Swaminathan [40].

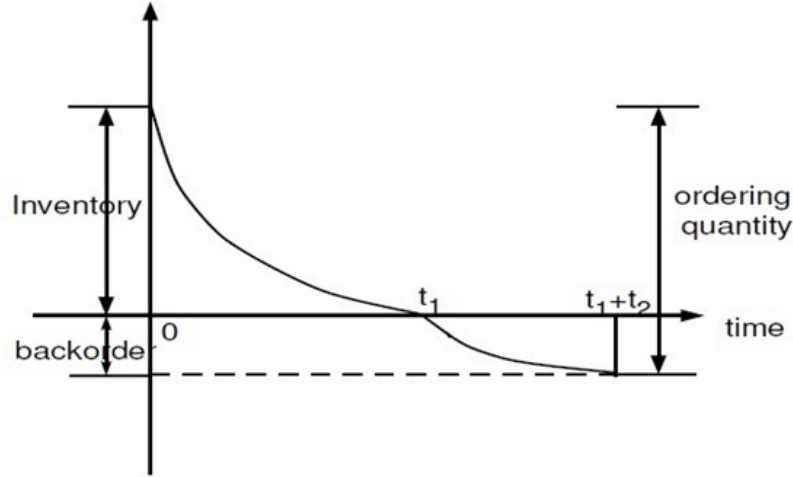
## 3. Model Formulation

According to the notation and assumptions mentioned above, the behavior of inventory system can be depicted in the following Figure 1. It can be seen that the depletion of the inventory occurs due to the combined effects of demand and deterioration during the interval  $[0, t_1)$  and the accumulation of backlogging occurs due to the demand partially backlogged

during the interval  $[t_1, t_1 + t_2]$ . Then, from (1), the variation of inventory,  $I(t)$ , with respect to time can be described as

$$\frac{dI(t)}{dt} = -\alpha e^{-\mu t} - \theta I(t), \quad 0 \leq t \leq t_1, \quad (3)$$

with boundary condition  $I(t_1) = 0$ . Solving equation (3) about  $I(t)$ , we can derive



**Figure 1. The Development of the Inventory Level during a Cycle**

$$I(t) = e^{-\theta t} \int_t^{t_1} \alpha e^{(\theta-\mu)s} ds = \begin{cases} \alpha(t_1 - t)e^{-\theta t} & \theta = \mu, \\ \frac{\alpha}{\theta - \mu} [e^{-\theta t + (\theta-\mu)t_1} - e^{-\mu t}] & \theta \neq \mu, \end{cases} \quad (4)$$

which implies

$$I(0) = \begin{cases} \alpha t_1 & \theta = \mu, \\ \frac{\alpha}{\theta - \mu} [e^{(\theta-\mu)t_1} - 1] & \theta \neq \mu, \end{cases} \quad (5)$$

where  $I(0)$  denotes the initial inventory level and can be seen as the order-up-to level. From (2), the variation of backlogging,  $W(t)$ , with respect to time can be described by the following differential equation:

$$\frac{dW(t)}{dt} = \alpha - \gamma W(t), \quad t_1 \leq t \leq t_1 + t_2 \quad (6)$$

with boundary condition  $W(t_1) = 0$ . The solution of (6) can be represented by

$$W(t) = \frac{\alpha}{\gamma} [1 - e^{-(t-t_1)\gamma}], \quad t_1 \leq t \leq t_1 + t_2 \quad (7)$$

From (7), the cumulative lost sales at time  $t_1 + t_2$  can be described as

$$L(t_1 + t_2) = \alpha t_2 - \frac{\alpha}{\gamma} + \frac{\alpha}{\gamma} e^{-\gamma t_2}. \quad (8)$$

For analyzing simply, we define  $W_1(u) = W(t_1 + u)$  for  $0 \leq u \leq t_2$ . Therefore, for  $t_1 \leq t \leq t_1 + t_2$ , the backlogging,  $W(t)$ , can be described as the following expression equivalently:

$$W(t_1 + u) = W_1(u) = \frac{\alpha}{\gamma} (1 - e^{-\gamma u}), \quad 0 \leq u \leq t_2. \quad (9)$$

Thus, the sum of the shortage cost and opportunity cost due to lost sales in shortage interval  $[t_1, t_1 + t_2]$  can be formulated as:

$$c_1 \int_0^{t_2} \frac{\alpha}{\gamma} (1 - e^{-\gamma u}) du + c_2 \left( \alpha t_2 - \frac{\alpha}{\gamma} + \frac{\alpha}{\gamma} e^{-\gamma t_2} \right) \quad (10)$$

The first part of (10) denotes the shortage cost during the interval  $[t_1, t_1 + t_2]$ , and the second part of (10) denotes the opportunity cost due to lost sales. Similar to the description of costs, the sales revenue per cycle can be formulated as:

$$p \left[ \int_0^{t_1} \alpha e^{-\mu t} dt + W_1(t_2) \right] = p \left[ \frac{\alpha}{\mu} (1 - e^{-\mu t_1}) + \frac{\alpha}{\gamma} (1 - e^{-\gamma t_2}) \right] \quad (11)$$

and the holding cost in the interval  $[0, t_1)$  can be formulated as

$$h \int_0^{t_1} I(t) dt = h \int_0^{t_1} e^{-\theta t} \int_t^{t_1} \alpha e^{(\theta - \mu)s} ds dt. \quad (12)$$

Conjunct with the relevant costs and revenue, the profit per unit time can be described as

$$P(t_1, t_2) = \frac{1}{t_1 + t_2} \left\{ \begin{array}{l} \frac{\alpha p}{\mu} (1 - e^{-\mu t_1}) - c \int_0^{t_1} \alpha e^{(\theta - \mu)s} ds + \frac{\alpha}{\gamma} (p - c) (1 - e^{-\gamma t_2}) \\ - h \int_0^{t_1} e^{-\theta t} \int_t^{t_1} \alpha e^{(\theta - \mu)s} ds dt - c_1 \int_0^{t_2} \frac{\alpha}{\gamma} (1 - e^{-\gamma u}) du \\ - c_2 \left( \alpha t_2 - \frac{\alpha}{\gamma} + \frac{\alpha}{\gamma} e^{-\gamma t_2} \right) - K \end{array} \right\}. \quad (13)$$

For notational convenience, we define

$$\begin{aligned}
 g(t_2) &= \frac{\alpha}{\gamma}(p-c)(1-e^{-\gamma t_2}) - c_1 \int_0^{t_2} \frac{\alpha}{\gamma}(1-e^{-\gamma u}) du - c_2 \left( \alpha t_2 - \frac{\alpha}{\gamma} + \frac{\alpha}{\gamma} e^{-\gamma t_2} \right) \\
 &= \left[ \frac{\alpha}{\gamma}(p-c) + \frac{\alpha c_1}{\gamma^2} + \frac{\alpha c_2}{\gamma} \right] (1-e^{-\gamma t_2}) - \frac{\alpha c_1}{\gamma} t_2 - \alpha c_2 t_2. \tag{14}
 \end{aligned}$$

$$\begin{aligned}
 f(t_1) &= \frac{\alpha p}{\mu}(1-e^{-\mu t_1}) - c \int_0^{t_1} \alpha e^{(\theta-\mu)s} ds - h \int_0^{t_1} e^{-\theta t} \int_t^{t_1} \alpha e^{(\theta-\mu)s} ds dt \\
 &= \begin{cases} \frac{\alpha p}{\mu} + \frac{\alpha h}{\mu^2} - \left( \frac{\alpha p}{\mu} + \frac{\alpha h}{\mu^2} \right) e^{-\mu t_1} - \left( \alpha c + \frac{\alpha h}{\mu} \right) t_1 & \theta = \mu, \\ \frac{\alpha p}{\mu} + \frac{\alpha h}{\mu(\theta-\mu)} + \frac{\alpha c}{\theta-\mu} - \left( \frac{\alpha c}{\theta-\mu} + \frac{\alpha h}{\theta(\theta-\mu)} \right) e^{(\theta-\mu)t_1} & \theta \neq \mu. \\ - \left( \frac{\alpha p}{\mu} + \frac{\alpha h}{\theta\mu} \right) e^{-\mu t_1} & \end{cases} \tag{15}
 \end{aligned}$$

Taking derivatives of  $f(t_1)$  and  $g(t_2)$  with respect to  $t_1$  and  $t_2$ , respectively, we have

$$f'(t_1) = \left( \alpha p + \frac{\alpha h}{\theta} \right) e^{-\mu t_1} - \left( \alpha c + \frac{\alpha h}{\theta} \right) e^{(\theta-\mu)t_1}, \tag{16}$$

$$f''(t_1) = -\mu \left( \alpha p + \frac{\alpha h}{\theta} \right) e^{-\mu t_1} - (\theta - \mu) \left( \alpha c + \frac{\alpha h}{\theta} \right) e^{(\theta-\mu)t_1}, \tag{17}$$

$$g'(t_2) = \left[ \alpha(p-c) + \frac{\alpha c_1}{\gamma} + \alpha c_2 \right] e^{-\gamma t_2} - \frac{\alpha c_1}{\gamma} - \alpha c_2, \tag{18}$$

$$g''(t_2) = -\gamma \left[ \alpha(p-c) + \frac{\alpha c_1}{\gamma} + \alpha c_2 \right] e^{-\gamma t_2} < 0. \tag{19}$$

In accordance with expression (13), the retailer's objective is to maximize the profit per unit time,  $P(t_1, t_2)$ , by choosing a pair of optimal  $(t_1^*, t_2^*)$ , which can be described as

$$P^* = \max_{t_1 \geq 0, t_2 \geq 0} \left\{ P(t_1, t_2) = \frac{f(t_1) + g(t_2) - K}{t_1 + t_2} \right\}. \tag{20}$$

In expression (20), we define  $P(0,0) = 0$ , which means that the retailer's profit will be zero if he/she neither orders nor sells the item. From (14) and (15), we can derive

$$\lim_{t_1 \rightarrow 0} f(t_1) = \lim_{t_2 \rightarrow 0} g(t_2) = 0, \quad \lim_{t_1 \rightarrow \infty} f(t_1) \leq \frac{\alpha p}{\mu} + \frac{\alpha h}{\mu(\theta-\mu)}$$

and

$$\lim_{t_2 \rightarrow \infty} g(t_2) \leq \frac{\alpha}{\gamma}(p-c) + \frac{\alpha c_1}{\gamma^2} + \frac{\alpha c_2}{\gamma}.$$

Therefore, for any fixed  $\bar{t}_1$  and  $\bar{t}_2$ , we have

$$\lim_{t_1 \rightarrow \infty} P(t_1, \bar{t}_2) \leq 0, \quad \lim_{t_2 \rightarrow \infty} P(\bar{t}_1, t_2) \leq 0 \quad \text{and} \quad \lim_{t_1 \rightarrow \infty, t_2 \rightarrow \infty} P(t_1, t_2) \leq 0.$$

Then the optimal solution,  $(t_1^*, t_2^*)$ , not only exists but also is finite, i.e.,  $t_1^* < \infty$  and  $t_2^* < \infty$ . From expression (20), we can conclude that there are four cases for the optimal solution,  $(t_1^*, t_2^*)$ . The first case is  $0 < t_1^* < \infty$  and  $0 < t_2^* < \infty$ ; the second case is  $0 < t_1^* < \infty$  and  $t_2^* = 0$ ; the third case is  $t_1^* = 0$  and  $0 < t_2^* < \infty$ ; the fourth case is  $t_1^* = t_2^* = 0$ . From these four cases, we see that the fourth case is an extreme case, under which the retailer will not order and sell the perishable item. In order to eliminate this trivial case in this study, we first present the following two results which are useful to abandon the fourth case.

**Lemma 3.1**

$$A \doteq \max_{t_1 \geq 0} \{f(t_1)\} = \begin{cases} \frac{\alpha}{\mu}(p-c) - \frac{1}{\mu} \left( \alpha c + \frac{\alpha h}{\theta} \right) \ln \frac{p + \frac{h}{\theta}}{c + \frac{h}{\theta}} & \theta = \mu, \\ \frac{\alpha p}{\mu} + \frac{\alpha c}{\theta - \mu} + \frac{\alpha h}{\mu(\theta - \mu)} - \frac{\theta}{\mu(\theta - \mu)} \left( \alpha p + \frac{\alpha h}{\theta} \right) \left( \frac{c + \frac{h}{\theta}}{p + \frac{h}{\theta}} \right)^{\frac{\mu}{\theta}} & \theta \neq \mu. \end{cases} \quad (21)$$

**Proof.** From (16) and (17), the solution to  $f'(t_1) = 0$  can be described as

$$t_0 = \frac{1}{\theta} \ln \frac{p + \frac{h}{\theta}}{c + \frac{h}{\theta}} > 0 \quad \text{and} \quad f''(t_1) \Big|_{f'(t_1)=0} = -\theta \left( \alpha p + \frac{\alpha h}{\theta} \right) e^{-\mu t_0} < 0.$$

Then  $t_0 = \frac{1}{\theta} \ln \frac{p + \frac{h}{\theta}}{c + \frac{h}{\theta}}$  is the unique maximizer of function  $f(t_1)$  on interval  $[0, \infty)$ .

Substituting  $t_0 = \frac{1}{\theta} \ln \frac{p + \frac{h}{\theta}}{c + \frac{h}{\theta}}$  into (15), we have that if  $\theta = \mu$ , then

$$\max_{t_1 \geq 0} \{f(t_1)\} = \frac{\alpha}{\mu}(p-c) - \frac{1}{\mu} \left( \alpha c + \frac{\alpha h}{\theta} \right) \ln \frac{p + \frac{h}{\theta}}{c + \frac{h}{\theta}},$$

and if  $\theta \neq \mu$ , then

$$\max_{t_1 \geq 0} \{f(t_1)\} = \frac{\alpha p}{\mu} + \frac{\alpha c}{\theta - \mu} + \frac{\alpha h}{\mu(\theta - \mu)} - \frac{\theta}{\mu(\theta - \mu)} \left( \alpha p + \frac{\alpha h}{\theta} \right) \left( \frac{c + \frac{h}{\theta}}{p + \frac{h}{\theta}} \right)^{\frac{\mu}{\theta}}.$$

The proof is completed.

Similar to Lemma 3.1, we derive the following result which formulates the maximal profit the retailer can obtain during the interval of shortage.

**Lemma 3.2**

$$B \doteq \max_{t_2 \geq 0} \{g(t_2)\} = \frac{\alpha}{\gamma} (p - c) - \frac{1}{\gamma} \left( \frac{\alpha c_1}{\gamma} + \alpha c_2 \right) \ln \frac{\alpha(p - c) + \frac{\alpha c_1}{\gamma} + \alpha c_2}{\frac{\alpha c_1}{\gamma} + \alpha c_2}. \quad (22)$$

**Proof.** From (19), we get that the function  $g(t_2)$  is strictly concave in  $t_2$  on interval  $[0, \infty)$ . Hence, the solution to equation (18),

$$t_2^0 = \frac{1}{\gamma} \ln \frac{\alpha(p - c) + \frac{\alpha c_1}{\gamma} + \alpha c_2}{\frac{\alpha c_1}{\gamma} + \alpha c_2}$$

is the unique maximizer of  $g(t_2)$  on  $[0, \infty)$ . Substituting  $t_2^0$  into expression (14) and simplifying, we have

$$\max_{t_2 \geq 0} \{g(t_2)\} = g(t_2^0) = \frac{\alpha}{\gamma} (p - c) - \frac{1}{\gamma} \left( \frac{\alpha c_1}{\gamma} + \alpha c_2 \right) \ln \frac{\alpha(p - c) + \frac{\alpha c_1}{\gamma} + \alpha c_2}{\frac{\alpha c_1}{\gamma} + \alpha c_2}.$$

The proof is completed.

Following above two Lemmas and expression (20), we give the following assumption under which the fourth case can not occur in the optimal situation.

**Assumption 3.1**

$$\max_{t_1 \geq 0} \{f(t_1)\} + \max_{t_2 \geq 0} \{g(t_2)\} - K > 0,$$

where  $\max_{t_1 \geq 0} \{f(t_1)\}$  and  $\max_{t_2 \geq 0} \{g(t_2)\}$  can be obtained by (21) and (22), respectively.

Assumption 3.1 formulates the necessary and sufficient condition under which the retailer will prefer to ordering the item and selling them, that is, if Assumption 3.1 is not satisfied, then it is better for the retailer not to sell the item during the sales horizon because the profit is always nonpositive. Hence, throughout this study, we assume that Assumption 3.1 always holds. In the rest of this section, we mainly discuss the necessary conditions which the optimal solution satisfies in the first case. Because the optimal solution of the first case satisfies  $(t_1^*, t_2^*) \in (0, \infty) \times (0, \infty)$ , the following first order necessary condition must be satisfied:

$$P'_{t_1}(t_1, t_2) |_{(t_1^*, t_2^*)} = 0 \text{ and } P'_{t_2}(t_1, t_2) |_{(t_1^*, t_2^*)} = 0.$$



By calculating, the necessary condition can be written as

$$P'_{t_1}(t_1, t_2) \Big|_{(t_1^*, t_2^*)} = \frac{f'(t_1^*)(t_1^* + t_2^*) - [f(t_1^*) + g(t_2^*) - K]}{(t_1^* + t_2^*)^2} = 0,$$

$$P'_{t_2}(t_1, t_2) \Big|_{(t_1^*, t_2^*)} = \frac{g'(t_2^*)(t_1^* + t_2^*) - [f(t_1^*) + g(t_2^*) - K]}{(t_1^* + t_2^*)^2} = 0.$$

By simplifying, the necessary condition is equivalent to the following expressions:

$$f'(t_1^*) - g'(t_2^*) = 0 \quad \text{and} \quad f(t_1^*) + g(t_2^*) - f'(t_1^*)(t_1^* + t_2^*) - K = 0. \quad (23)$$

Whether the necessary condition (23) is also sufficient is the most important problem in this study, which will be answered in detail in the following section. Moreover, we can show that the second and the third cases will not occur in the optimal situation. Hence, the optimal solution for the first case is also the optimal solution for the retailer's objective function (13). All of these results will be formulated and proved in the following section.

#### 4. The Optimal Solution

In the above section, we have formulated an EOQ model for perishable item with freshness-dependent demand. In this section, we will concentrate on analyzing the optimal solution of the profit function,  $P(t_1, t_2)$ , defined as (13). In what follows, we first prove that the optimal solution,  $(t_1^*, t_2^*)$ , must satisfy  $t_1^* > 0$  and  $t_2^* > 0$ , where  $(t_1^*, t_2^*)$  is the solution of expression (20) on  $[0, \infty) \times [0, \infty)$ . That is, in the optimal situation, both the sale of item and the shortage of item always occur. In order to prove  $t_1^* > 0$  and  $t_2^* > 0$ , we first present the following results which are easy but useful.

**Lemma 4.1** For any  $t_1 > 0$  and  $t_2 > 0$ ,

$$\frac{f(t_1) + g(t_2)}{t_1 + t_2} > \frac{f(t_1)}{t_1} \quad \text{if and only if} \quad \frac{g(t_2)}{t_2} > \frac{f(t_1)}{t_1}$$

**Proof.** Note that  $t_1 > 0$  and  $t_2 > 0$ , then

$$\frac{f(t_1) + g(t_2)}{t_1 + t_2} > \frac{f(t_1)}{t_1} \quad \text{is equivalent to} \quad [f(t_1) + g(t_2)]t_1 > f(t_1)(t_1 + t_2).$$

By simplifying,  $[f(t_1) + g(t_2)]t_1 > f(t_1)(t_1 + t_2)$  can be written as  $g(t_2)t_1 > f(t_1)t_2$ ,

which is equivalent to  $\frac{g(t_2)}{t_2} > \frac{f(t_1)}{t_1}$ . Therefore,

$$\frac{f(t_1) + g(t_2)}{t_1 + t_2} > \frac{f(t_1)}{t_1} \quad \text{is equivalent to} \quad \frac{g(t_2)}{t_2} > \frac{f(t_1)}{t_1}.$$

**Lemma 4.2** If  $\theta \geq \mu$ , then  $f(t_1)$  is strictly concave in  $t_1$ ; if  $\theta < \mu$ , then there exists unique  $\hat{t}_1$  such that  $f(t_1)$  is strictly concave on  $[0, \hat{t}_1)$  and strictly convex on  $(\hat{t}_1, \infty)$ , where

$$\hat{t}_1 = \frac{1}{\theta} \ln \frac{\mu \left( p + \frac{h}{\theta} \right)}{(\mu - \theta) \left( c + \frac{h}{\theta} \right)}.$$

**Proof.** From (17), we have

$$f''(t_1) = \left[ -\mu \left( \alpha p + \frac{\alpha h}{\theta} \right) - (\theta - \mu) \left( \alpha c + \frac{\alpha h}{\theta} \right) e^{\theta t_1} \right] e^{-\mu t_1}.$$

For notational convenience, we denote

$$\hat{f}(t_1) = -\mu \left( \alpha p + \frac{\alpha h}{\theta} \right) - (\theta - \mu) \left( \alpha c + \frac{\alpha h}{\theta} \right) e^{\theta t_1}.$$

If  $\theta \geq \mu$ , then  $f''(t_1) < 0$  and  $f(t_1)$  is strictly concave in  $t_1$ ; if  $\theta < \mu$ , then  $\hat{f}(t_1)$  is strictly increasing in  $t_1$  and there exists unique  $\hat{t}_1$  such that  $f''(t_1) = 0$ ,  $f''(t_1) < 0$  for  $0 \leq t_1 < \hat{t}_1$  and  $f''(t_1) > 0$  for  $t_1 > \hat{t}_1$ , where

$$\hat{t}_1 = \frac{1}{\theta} \ln \frac{\mu \left( p + \frac{h}{\theta} \right)}{(\mu - \theta) \left( c + \frac{h}{\theta} \right)} \geq \frac{1}{\theta} \ln \frac{\left( p + \frac{h}{\theta} \right)}{\left( c + \frac{h}{\theta} \right)} = t_0$$

and  $f'(t_0) = 0$ . That is, if  $\theta < \mu$ , then  $f(t_1)$  is strictly concave on  $[0, \hat{t}_1)$  and strictly convex on  $(\hat{t}_1, \infty)$ .

Based on Lemma 4.2 and Lemma 4.1, we can derive the following result which states that the optimal solution of profit function, defined as (13), belongs to the open set  $(0, \infty) \times (0, \infty)$ .

**Theorem 4.1** If Assumption 3.1 is satisfied, then we have  $t_1^* > 0$  and  $t_2^* > 0$ , where  $(t_1^*, t_2^*)$  is the optimal solution of profit function (13).

**Proof.** We will prove this result by contradiction. Assume  $t_1^* = 0$ , then from Assumption 3.1, we can derive  $t_2^* > 0$  and  $B > K$ . Note that  $t_2^* > 0$ ,  $t_1^* = 0$ ,  $B > K$  and  $f(0) = 0$  from (15), we obtain

$$\max_{t_1 \geq 0} \left\{ \frac{f(t_1) + g(\hat{t}_2^*) - K}{t_1 + \hat{t}_2^*} \right\} \leq \max_{t_1 \geq 0, t_2 \geq 0} \left\{ \frac{f(t_1) + g(t_2) - K}{t_1 + t_2} \right\} = \frac{g(\hat{t}_2^*) - K}{\hat{t}_2^*}, \quad (24)$$

$$\text{where } \hat{t}_2^* = \arg \max_{t_2 \geq 0} \left\{ \frac{g(t_2) - K}{t_2} \right\}.$$

Next, we will derive a result which is in contradiction with expression (24). Since  $g(0) = 0$  and  $g(t_2)$  is strictly concave in  $t_2$ , we have from (18) that

$$\frac{g(\hat{t}_2^*) - K}{\hat{t}_2^*} \leq \frac{g(\hat{t}_2^*) - g(0)}{\hat{t}_2^* - 0} < g'(0) = \alpha(p - c).$$

From (16), we have that  $f'(0) = \alpha(p - c) = g'(0)$  and  $f'(t_1)$  is continuous on  $[0, \infty)$ .

Therefore, there exists some positive number  $x \in (0, \hat{t}_1)$  such that

$$f'(x) > f'(0) - \frac{1}{2} \left[ g'(0) - \frac{g(\hat{t}_2^*) - K}{\hat{t}_2^*} \right] = \frac{1}{2} g'(0) + \frac{1}{2} \frac{g(\hat{t}_2^*) - K}{\hat{t}_2^*} > \frac{g(\hat{t}_2^*) - K}{\hat{t}_2^*}.$$

Since  $f(0) = 0$  and  $f(t_1)$  is strictly concave on  $[0, \hat{t}_1)$ , we derive

$$\frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} > f'(x) > \frac{g(\hat{t}_2^*) - K}{\hat{t}_2^*}.$$

Note that  $x > 0$  and  $\hat{t}_2^* > 0$ , then from Lemma 4.1, we have

$$\frac{f(x) + g(\hat{t}_2^*) - K}{x + \hat{t}_2^*} > \frac{g(\hat{t}_2^*) - K}{\hat{t}_2^*},$$

which is in contradiction to expression (24). Hence, we have  $t_1^* > 0$ . In Lemma 4.2, we have proved that  $f(t_1)$  is strictly concave on  $[0, \hat{t}_1)$ . Therefore, we can prove  $t_2^* > 0$  in a similar way.

Theorem 4.1 states that, under Assumption 3.1, the optimal solution on the open set  $(0, \infty) \times (0, \infty)$  is just the optimal solution on  $[0, \infty) \times [0, \infty)$ . Hence, we will discuss the optimal solution just for the first case,  $t_1^* > 0$  and  $t_2^* > 0$ , in the rest of this section. Moreover, we can obtain the maximizer of profit function  $P(t_1, t_2)$  on  $(0, \infty) \times (0, \infty)$  by solving the condition (23) directly. The following result formulates the optimal solution of the profit per unit time under condition  $\theta \neq \mu$ .

**Proposition 4.1** If  $\theta \neq \mu$ , then the optimal solution of profit per unit time,  $(t_1^*, t_2^*)$ , not only exists but also is unique on  $(0, \infty) \times (0, \infty)$ . Moreover, the optimal solution,  $(t_1^*, t_2^*)$ , can be obtained by solving the following equations:

$$\begin{aligned} & \frac{p}{\mu} + \frac{h}{\mu(\theta - \mu)} + \frac{c}{\theta - \mu} - \frac{K}{\alpha} - \left( \frac{h}{\theta(\theta - \mu)} + \frac{c}{\theta - \mu} \right) e^{(\theta - \mu)t_1^*} - \\ & \left( \frac{p}{\mu} + \frac{h}{\theta\mu} \right) e^{-\mu t_1^*} - \left[ \left( p + \frac{h}{\theta} \right) e^{-\mu t_1^*} - \left( c + \frac{h}{\theta} \right) e^{(\theta - \mu)t_1^*} \right] t_1^* + \frac{1}{\gamma} (p - c) + \frac{c_1}{\gamma^2} \\ & + \frac{c_2}{\gamma} - \frac{1}{\gamma} \left[ \left( p + \frac{h}{\theta} \right) e^{-\mu t_1^*} - \left( c + \frac{h}{\theta} \right) e^{(\theta - \mu)t_1^*} + \frac{c_1}{\gamma} + c_2 \right] \\ & \left\{ 1 + \ln \frac{p - c + \frac{c_1}{\gamma} + c_2}{\left( p + \frac{h}{\theta} \right) e^{-\mu t_1^*} - \left( c + \frac{h}{\theta} \right) e^{(\theta - \mu)t_1^*} + \frac{c_1}{\gamma} + c_2} \right\} = 0, \end{aligned} \quad (25)$$

$$t_2^* = \frac{1}{\gamma} \ln \frac{p - c + \frac{c_1}{\gamma} + c_2}{\left( p + \frac{h}{\theta} \right) e^{-\mu t_1^*} - \left( c + \frac{h}{\theta} \right) e^{(\theta - \mu)t_1^*} + \frac{c_1}{\gamma} + c_2}. \quad (26)$$

**Proof.** Firstly, we prove this result for the case of  $\theta > \mu$ . From Theorem 4.1 and expression (23), the optimal solution,  $(t_1^*, t_2^*)$ , not only satisfies  $0 < t_1^* < \infty$  and  $0 < t_2^* < \infty$ , but also is one solution to the first order necessary condition:

$$f'(t_1^*) - g'(t_2^*) = 0 \quad \text{and} \quad f(t_1^*) + g(t_2^*) - f'(t_1^*)(t_1^* + t_2^*) - K = 0. \quad (27)$$

Since the optimal solution always exists and be finite, there must be finite solutions to equations (27). Note that equations (27) are just necessary condition, therefore, some finite solutions to equations (27) are not necessarily optimal. However, if there is only one solution to equations (27), then the unique solution must be optimal. Next, we want to prove that equations (27) has only one solution. From (16) and (18),  $f'(t_1) = g'(t_2)$  can be written as

$$\left( p + \frac{h}{\theta} \right) e^{-\mu t_1} - \left( c + \frac{h}{\theta} \right) e^{(\theta - \mu)t_1} = \left( p - c + \frac{c_1}{\gamma} + c_2 \right) e^{-\mu t_2} - \frac{c_1}{\gamma} - c_2. \quad (28)$$

By simplifying expression (28), one obtains

$$t_2(t_1) = \frac{1}{\gamma} \ln \frac{p - c + \frac{c_1}{\gamma} + c_2}{\left( p + \frac{h}{\theta} \right) e^{-\mu t_1} - \left( c + \frac{h}{\theta} \right) e^{(\theta - \mu)t_1} + \frac{c_1}{\gamma} + c_2}. \quad (29)$$

Note that  $\theta > \mu$ , then  $t_2(t_1)$  is strictly increasing in  $t_1$ . Substituting (29) into the second equation of (23), we have

$$f(t_1) + g(t_2(t_1)) - f'(t_1)(t_1 + t_2(t_1)) - K = 0.$$

For notational convenience, we define

$$G(t_1) = f(t_1) + g(t_2(t_1)) - f'(t_1)(t_1 + t_2(t_1)) - K. \quad (30)$$

Taking derivative of  $G(t_1)$  with respect to  $t_1$ , we derive the following expression:

$$\begin{aligned} G'(t_1) &= f'(t_1) + g'(t_2(t_1))t_2'(t_1) - f''(t_1)(t_1 + t_2(t_1)) - f'(t_1)(1 + t_2'(t_1)) \\ &= -f''(t_1)(t_1 + t_2(t_1)) > 0. \end{aligned}$$

The second equality is due to  $f'(t_1) = g'(t_2)$  and the inequality is due to Lemma 4.2. Therefore,  $G(t_1)$  is strictly increasing in  $t_1$  and then  $G(t_1) = 0$  has at most one solution. Since there must exist solutions to equations (27), the optimal solution  $t_1^*$  is the unique solution to  $G(t_1) = 0$ . Substituting (29), (14), (15) and (16) into (30) and simplifying, we have

$$\begin{aligned} G(t_1) &= \frac{p}{\mu} + \frac{h}{\mu(\theta - \mu)} + \frac{c}{\theta - \mu} - \frac{K}{\alpha} - \left( \frac{h}{\theta(\theta - \mu)} + \frac{c}{\theta - \mu} \right) e^{(\theta - \mu)t_1} - \\ &\left( \frac{p}{\mu} + \frac{h}{\theta\mu} \right) e^{-\mu t_1} - \left[ \left( p + \frac{h}{\theta} \right) e^{-\mu t_1} - \left( c + \frac{h}{\theta} \right) e^{(\theta - \mu)t_1} \right] t_1 + \frac{1}{\gamma} (p - c) + \frac{c_1}{\gamma^2} \\ &+ \frac{c_2}{\gamma} - \frac{1}{\gamma} \left[ \left( p + \frac{h}{\theta} \right) e^{-\mu t_1} - \left( c + \frac{h}{\theta} \right) e^{(\theta - \mu)t_1} + \frac{c_1}{\gamma} + c_2 \right] \\ &\left\{ 1 + \ln \frac{p - c + \frac{c_1}{\gamma} + c_2}{\left( p + \frac{h}{\theta} \right) e^{-\mu t_1} - \left( c + \frac{h}{\theta} \right) e^{(\theta - \mu)t_1} + \frac{c_1}{\gamma} + c_2} \right\}. \quad (31) \end{aligned}$$

Therefore, the optimal solution,  $(t_1^*, t_2^*)$ , satisfies equations (25) and (26).

Secondly, we prove the result for the case of  $\theta < \mu$ . In Lemma 4.2, we have shown that  $f''(t_1) < 0$  for  $0 \leq t_1 < \hat{t}_1$  and  $f''(t_1) > 0$  for  $t_1 > \hat{t}_1$ . Therefore, when  $\theta < \mu$ ,  $G(t_1)$  is strictly increasing on  $[0, \hat{t}_1)$  and strictly decreasing on  $(\hat{t}_1, \infty)$ . That is,  $\hat{t}_1$  is the unique maximizer of  $G(t_1)$  on interval  $[0, \infty)$ . Substituting  $t_1 = 0$  into (31) and simplifying, we have  $G(0) = -\frac{K}{\alpha} < 0$ .

Because the optimal solution always exists and satisfies  $G(t_1^*) = 0$ , we can derive  $G(\hat{t}_1) > 0$ . Therefore, on the interval  $(0, \hat{t}_1)$ , there must exist a unique solution  $t_1^*$  such

that  $G(t_1^*) = 0$ . Next, we want to prove that  $t_1^*$  is the unique optimal solution, that is,  $(t_1^*, t_2(t_1^*))$  is the unique optimal solution of the profit per unit time. From (31), we have

$$\lim_{t_1 \rightarrow \infty} G(t_1) = \frac{p}{\mu} + \frac{h}{\mu(\theta - \mu)} + \frac{c}{\theta - \mu} - \frac{K}{\alpha} + \frac{1}{\gamma}(p - c) + \frac{c_1}{\gamma^2} + \frac{c_2}{\gamma} - \frac{1}{\gamma} \left( \frac{c_1}{\gamma} + c_2 \right) \left\{ 1 + \ln \frac{p - c + \frac{c_1}{\gamma} + c_2}{\frac{c_1}{\gamma} + c_2} \right\}.$$

If  $\lim_{t_1 \rightarrow \infty} G(t_1) \geq 0$ , then  $t_1^*$  is the unique solution to  $G(t_1) = 0$  on interval  $(0, \infty)$ .

Therefore,  $t_1^*$  is the unique optimal solution, that is,  $(t_1^*, t_2(t_1^*))$  is the unique optimal solution of the profit per unit time. If  $\lim_{t_1 \rightarrow \infty} G(t_1) < 0$ , then there exists another solution  $t_1^0 > \hat{t}_1$  such that  $G(t_1^0) = 0$ . Next, we will prove that  $t_1^0$  can not be optimal. From (30), one obtains

$$G(t_1) = [t_1 + t_2(t_1)] \left[ \frac{f(t_1) + g(t_2(t_1)) - K}{t_1 + t_2(t_1)} - f'(t_1) \right].$$

Note that  $G(t_1^0) = 0$  and  $t_1^0 + t_2(t_1^0) > 0$ , we have

$$\frac{f(t_1^0) + g(t_2(t_1^0)) - K}{t_1^0 + t_2(t_1^0)} - f'(t_1^0) = 0. \quad (32)$$

From (16) and Lemma 3.1, we can derive  $f'(t_1) > 0$  for  $0 \leq t_1 < t_0$  and  $f'(t_1) < 0$  for  $t_1 > t_0$ , where

$$t_0 = \frac{1}{\theta} \ln \frac{p + \frac{h}{\theta}}{c + \frac{h}{\theta}}$$

is the unique maximizer of  $f(t_1)$  on interval  $(0, \infty)$ . From the proof of Lemma 4.2, we have  $\hat{t}_1 \geq t_0$ . Therefore, we can derive  $t_1^0 > t_0$  and  $f'(t_1^0) < 0$ . From (32), we have

$$\frac{f(t_1^0) + g(t_2(t_1^0)) - K}{t_1^0 + t_2(t_1^0)} = f'(t_1^0) < 0.$$

Thus,  $t_1^0$  will not be the optimal solution. Then  $(t_1^*, t_2(t_1^*))$  is the unique optimal solution of the profit per unit time.

Proposition 4.1 states that the unique optimal solution of the profit per unit time can be obtained by solving two equations. The next result formulates the optimal solution of the profit per unit time under condition  $\theta \neq \mu$ .

**Proposition 4.2** If  $\theta = \mu$ , then the optimal solution of profit per unit time,  $(t_1^*, t_2^*)$ , not only exists but also is unique on  $(0, \infty) \times (0, \infty)$ . Moreover, the optimal solution  $(t_1^*, t_2^*)$  satisfies the following equations:

$$t_2^* = \frac{1}{\gamma} \ln \frac{p - c + \frac{c_1}{\gamma} + c_2}{\left(p + \frac{h}{\theta}\right) e^{-\mu t_1^*} - c - \frac{h}{\theta} + \frac{c_1}{\gamma} + c_2}. \quad (33)$$

$$0 = \frac{p}{\mu} + \frac{h}{\mu^2} - \frac{K}{\alpha} - \left(\frac{p}{\mu} + \frac{h}{\mu^2}\right) e^{-\mu t_1^*} - \left(p + \frac{h}{\theta}\right) t_1^* e^{-\alpha t_1^*} + \frac{1}{\gamma} (p - c) + \frac{c_1}{\gamma^2} + \frac{c_2}{\gamma} - \frac{1}{\gamma} \left[ \left(p + \frac{h}{\theta}\right) e^{-\mu t_1^*} - c - \frac{h}{\theta} + \frac{c_1}{\gamma} + c_2 \right] \left\{ 1 + \ln \frac{p - c + \frac{c_1}{\gamma} + c_2}{\left(p + \frac{h}{\theta}\right) e^{-\mu t_1^*} - c - \frac{h}{\theta} + \frac{c_1}{\gamma} + c_2} \right\}. \quad (34)$$

**Proof.** When  $\theta = \mu$ , we derive from (17) that  $f(t_1)$  is strictly concave in  $t_1$ . From (16) and (18),  $f'(t_1) = g'(t_2)$  can be written as

$$\left(p + \frac{h}{\theta}\right) e^{-\mu t_1} - \left(c + \frac{h}{\theta}\right) = \left(p - c + \frac{c_1}{\gamma} + c_2\right) e^{-\mu t_2} - \frac{c_1}{\gamma} - c_2. \quad (35)$$

By simplifying expression (35), one obtains

$$t_2(t_1) = \frac{1}{\gamma} \ln \frac{p - c + \frac{c_1}{\gamma} + c_2}{\left(p + \frac{h}{\theta}\right) e^{-\mu t_1} - c - \frac{h}{\theta} + \frac{c_1}{\gamma} + c_2}.$$

Similar to the proof of Proposition 4.1,  $G(t_1)$  is strictly increasing in  $t_1$  and there is only one solution,  $t_1^*$ , such that  $G(t_1^*) = 0$ , where  $G(t_1)$  is defined as (30). Moreover, the optimal solution of profit per unit time,  $(t_1^*, t_2^*)$ , not only exists but also is unique on  $(0, \infty) \times (0, \infty)$ , and  $(t_1^*, t_2^*)$  can be obtained by solving equations (33) and (34).

Combining Proposition 4.1 with Proposition 4.2, the unique optimal solution of profit per unit time,  $(t_1^*, t_2^*)$ , can always be obtained by solving several equations.

### 5. Numerical Example

In the previous section, we have analyzed the optimal solution of the profit function per unit time and obtained the equations that the optimal solution satisfies. In this section, we will mainly study the effects of the factors,  $\mu$ ,  $\theta$ ,  $\gamma$  and  $c_1$ , on the optimal solution  $(t_1^*, t_2^*)$ , the optimal profit  $P^*$ , the optimal order-up-to level for each cycle,  $I^* \doteq I(0)$ , the corresponding wastage during a cycle,  $W^* \doteq I(0) - \int_0^{t_1^*} \alpha e^{-\mu t} dt$ ,

the cumulative backlogging during a cycle,  $B^* \doteq W_1(t_2^*)$ , the cumulative lost sales during a cycle,  $L^* \doteq L(t_1^* + t_2^*)$ , and the optimal ordering quantity for each cycle,  $Q^* \doteq I(0) + W_1(t_2^*)$ , by the following numerical examples.

In Table 1, we present the effect of  $\mu$  on the optimal solution, the optimal profit, the optimal order-up-to level for each cycle, the corresponding wastage during a cycle, the cumulative backlogging during a cycle, the cumulative lost sales during a cycle and the optimal ordering quantity. From Table 1, we can see that  $t_1^*$ ,  $P^*$ ,  $Q^*$ ,  $I^*$  and  $W^*$  are all decreasing in  $\mu$ , but  $t_2^*$ ,  $B^*$  and  $L^*$  are all increasing in  $\mu$ . Such phenomenon implies that the faster the freshness of the item reduces, the less the optimal ordering quantity for each cycle, the wastage during a cycle and the optimal profit per unit time are, the lower the optimal order-up-level for each cycle is, but the more the cumulative backlogging during a cycle and the cumulative lost sales during a cycle are. Therefore, the retailer always hopes that the freshness of the item reduces slowly, that is, the smaller the  $\mu$ , the better the retailer.

**Table 1. Effect of  $\mu$  on the Optimal Decisions and Profit**

Parameters :  $K = 250$ ,  $\alpha = 60$ ,  $\theta = 0.06$ ,  $p = 8$ ,  $c = 5$ ,  $c_1 = 4$ ,  $c_2 = 3$ ,  $h = 0.3$ ,  $\gamma = 0.2$ .

$\mu$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
$t_1^*$	3.2905	3.2796	3.2796	3.2686	3.2686	3.2686	3.2577	3.2577	3.2577	3.2577
$t_2^*$	0.4440	0.4477	0.4530	0.4565	0.4615	0.4663	0.4695	0.4740	0.4784	0.4827
$P^*$	47.9002	46.566	45.258	43.977	42.722	41.491	40.286	39.104	37.945	36.809
$Q^*$	240.089	235.960	232.763	228.848	225.808	222.83	219.2	216.365	213.589	210.869
$I^*$	214.599	210.263	206.779	202.67	199.358	196.117	192.312	189.23	186.214	183.262
$W^*$	20.382	19.804	19.376	18.832	18.428	18.033	17.535	17.163	16.8	16.445
$B^*$	25.49	25.697	25.985	26.179	26.45	26.713	26.888	27.135	27.375	27.607
$L^*$	1.1484	1.1677	1.1949	1.2133	1.2394	1.2649	1.2821	1.3066	1.3306	1.354

In Table 2, we present the effect of  $\theta$  on the optimal solution, the optimal profit, the optimal order-up-to level for each cycle, the corresponding wastage during a cycle, the cumulative backlogging during a cycle, the cumulative lost sales during a cycle and the optimal ordering quantity. From Table 2, we can see that  $t_1^*$ ,  $P^*$ ,  $Q^*$  and  $I^*$  are all decreasing in  $\theta$ , but  $t_2^*$ ,  $W^*$ ,  $B^*$  and  $L^*$  are all increasing in  $\theta$ . Such phenomenon implies that the larger the wastage rate of the item is, the less the optimal ordering quantity for each cycle and the optimal profit per unit time are, the lower the optimal order-up-level for each cycle is, but the more the cumulative backlogging during a cycle, the wastage during a cycle



and the cumulative lost sales during a cycle are. Therefore, the smaller the  $\theta$ , the better the retailer.

**Table 2. Effect of  $\theta$  on the Optimal Decisions and Profit**

Parameters :  $K = 250, \alpha = 60, \mu = 0.05, p = 8, c = 5, c_1 = 4, c_2 = 3, h = 0.3, \gamma = 0.2.$

$\theta$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
$t_1^*$	4.3997	4.1055	3.8591	3.6300	3.4366	3.2686	3.1096	2.9756	2.8528	2.7307
$t_2^*$	0.3703	0.3899	0.4095	0.4268	0.4442	0.4615	0.4768	0.4930	0.5084	0.5213
$P^*$	68.836	63.109	57.653	52.45	47.479	42.722	38.164	33.79	29.589	25.549
$Q^*$	263.473	254.272	246.429	238.438	231.697	225.808	219.784	214.919	210.32	205.26
$I^*$	242.06	231.765	222.838	213.895	206.194	199.358	192.497	186.75	181.315	175.557
$W^*$	5.095	9.073	12.259	14.716	16.742	18.428	19.704	20.858	21.796	22.406
$B^*$	21.413	22.507	23.59	24.544	25.503	26.45	27.287	28.17	29.006	29.703
$L^*$	0.8026	0.889	0.9792	1.0623	1.1496	1.2394	1.3218	1.4117	1.4998	1.5753

In Table 3, we present the effect of  $\gamma$  on the optimal solution, the optimal profit, the optimal order-up-to level for each cycle, the corresponding wastage during a cycle, the cumulative backlogging during a cycle, the cumulative lost sales during a cycle and the optimal ordering quantity. From Table 3, we can see that  $t_2^*, P^*, Q^*$  and  $B^*$  are all decreasing in  $\gamma$ , but  $t_1^*, I^*, W^*$  and  $L^*$  are all increasing in  $\gamma$ . Such phenomenon implies that the larger the lost sales rate of the item is, the less the optimal ordering quantity for each cycle, the cumulative backlogging during a cycle and the optimal profit per unit time are, but the more the wastage during a cycle and the cumulative lost sales during a cycle are, the higher the optimal order-up-level for each cycle is. Therefore, the smaller the  $\gamma$ , the better the retailer.

**Table 3. Effect of  $\gamma$  on the Optimal Decisions and Profit**

Parameters :  $K = 250, \alpha = 60, \theta = 0.06, \mu = 0.08, p = 8, c = 5, c_1 = 4, c_2 = 3, h = 0.3.$

$\gamma$	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
$t_1^*$	3.2140	3.2249	3.2468	3.2577	3.2686	3.2796	3.2905	3.3014
$t_2^*$	0.5481	0.5202	0.4966	0.4740	0.4536	0.4350	0.4180	0.4024
$P^*$	40.693	40.108	39.5804	39.104	38.671	38.275	37.912	37.578
$Q^*$	219.21	217.797	217.33	216.365	215.573	214.926	214.402	213.986
$I^*$	186.771	187.386	188.615	189.23	189.844	190.459	191.073	191.687
$W^*$	16.729	16.837	17.054	17.163	17.272	17.382	17.492	17.602
$B^*$	32.44	30.412	28.714	27.135	25.728	24.467	23.33	22.299
$L^*$	0.4465	0.7978	1.0828	1.3066	1.4863	1.6311	1.748	1.8425

From Table 4, we can see that  $t_2^*, P^*, Q^*, B^*$  and  $L^*$  are all decreasing in  $c_1$ , but  $t_1^*, I^*$  and  $W^*$  are all increasing in  $c_1$ . Such phenomenon implies that the smaller the shortage cost, the less the optimal ordering quantity for each cycle, the cumulative backlogging during

a cycle, the cumulative lost sales during a cycle and the optimal profit per unit time, but the more the wastage during a cycle and the higher the optimal order-up-level for each cycle. Note that the shortage cost and the opportunity cost of lost sales have the same effects on the corresponding variables, therefore, the smaller the shortage cost or the opportunity cost of lost sales, the better the retailer.

**Table 4. Effect of  $c_1$  on the Optimal Decisions and Profit**

Parameters :  $K = 250$ ,  $\alpha = 60$ ,  $\theta = 0.06$ ,  $\mu = 0.08$ ,  $p = 8$ ,  $c = 5$ ,  $c_2 = 3$ ,  $h = 0.3$ ,  $\gamma = 0.2$ .

$c_1$	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4
$t_1^*$	3.2030	3.2140	3.2249	3.2249	3.2358	3.2358	3.2468	3.2468	3.2577	3.2577
$t_2^*$	0.5704	0.5584	0.5470	0.5344	0.5240	0.5125	0.5031	0.4925	0.4838	0.4740
$P^*$	40.982	40.738	40.504	40.280	40.064	39.858	39.658	39.467	39.2815	39.104
$Q^*$	218.501	218.473	218.473	217.797	217.851	217.229	217.33	216.755	216.897	216.365
$I^*$	186.156	186.771	187.386	187.386	188.001	188.001	188.615	188.615	189.23	189.23
$W^*$	16.621	16.729	16.837	16.837	16.945	16.945	17.054	17.054	17.163	17.163
$B^*$	32.345	31.702	31.087	30.412	29.851	29.229	28.714	28.14	27.667	27.135
$L^*$	1.8801	1.8033	1.7314	1.6542	1.5916	1.5237	1.4687	1.4086	1.3601	1.3066

## 6. Conclusions and Future Research

In this paper, an inventory model is developed for deteriorating items with freshness-dependent demand, permitting shortages and partial backlogging. In particular, the demand rate is considered to be decreasing function with the age of the perishable item since the consumers prefer to purchasing fresher items for given price. These two assumptions are rational in reality. We provide the conditions under which the optimal solution of profit per unit time not only exists but also is unique. Moreover, the optimal solution can be obtained by solving several equations.

Through the extensive numerical analysis, we study the effects of the factors,  $\mu$ ,  $\theta$ ,  $\gamma$  and  $c_1$ , on the optimal solution, the optimal profit, the optimal order-up-to level for each cycle, the corresponding wastage during a cycle, the cumulative backlogging during a cycle, the cumulative lost sales during a cycle and the optimal ordering quantity for each cycle. In this study, we consider deterministic demand, however, there is random factor affecting demand process in reality. Therefore, a potential future research is to study the EOQ model with time-varying stochastic demand, such as diffusion process.

## Acknowledgements

We thank the editor-in-chief Adrian Stoica and three anonymous referees for their valuable comments, which help us improve the paper greatly. This work was partially supported by Natural Science Foundation of China under grant 71201027, Humanity and Social Science Youth foundation of Ministry of Education of China under grant 12YJC630260, Guangdong Natural Science Foundation under Grand S2012040007919, foundation for Distinguished Young Talents in Higher Education of Guangdong under grand LYM11121 and the Open Fund of Chongqing Key Laboratory of Logistics under grand CQKLL12003.

## References

- [1] G. Padmanabhan and P. Vrat, "EOQ models for perishable items under stock dependent selling rate", *European Journal of Operational Research*, vol. 86, (1995), pp. 281-292.
- [2] K. J. Chung, P. Chu and S. P. Lan, "A note on EOQ models for deteriorating items under stock dependent selling rate", *European Journal of Operational Research*, vol. 124, (2000), pp. 550-559.
- [3] C. -Y. Dye and L. Y. Ouyang, "An EOQ model for perishable items under stock-dependent selling rate and time-dependent partial backlogging", *European Journal of Operational Research*, vol. 163, (2005), pp. 776-783.
- [4] J. -M. Chen, "An inventory model for deteriorating items with time-proportional demand and shortages under inflation and time discounting", *International Journal of Production Economics*, vol. 55, (1998), pp. 21-30.
- [5] K. Skouri and S. Papachristos, "A continuous review inventory model, with deteriorating items, time-varying demand, linear replenishment cost, partially time-varying backlogging", *Applied Mathematical Modelling*, vol. 26, (2002), pp. 603-617.
- [6] S. -P. Wang, "Optimal lot-sizing policy for time-varying demand with shortage and deterioration", *Journal of the Chinese Institute of Industrial Engineers*, vol. 20, (2003), pp. 449-456.
- [7] S. Papachristos and K. Skouri, "An optimal replenishment policy for deteriorating items with time-varying demand and partial-exponential type-backlogging", *Operations Research Letters*, vol. 27, (2000), pp. 175-184.
- [8] S. Papachristos and K. Skouri, "An inventory model with deteriorating items, quantity discount, pricing and time-dependent partial backlogging", *International Journal of Production Economics*, vol. 83, (2003), pp. 247-256.
- [9] W. -H. Lee and J. -W. Wu, "An EOQ model for items with Weibull distributed deterioration, shortages and Power demand pattern", *International Journal of Information and Management Sciences*, vol. 13, (2002), pp. 19-34.
- [10] W. -H. Lee and J. -W. Wu, "A note on EOQ model for items with mixtures of exponential distribution deterioration, shortages and time-varying demand", *Quality and Quantity*, vol. 38, (2004), pp. 457-473.
- [11] J. -W. Wu, C. Lin, B. Tan and W. -C. Lee, "An EOQ inventory model with time-varying demand and Weibull deterioration with shortages", *International Journal of Systems Science*, vol. 31, (2000), pp. 677-683.
- [12] J. Min and Y. -W. Zhou, "A perishable inventory model under stock-dependent selling rate and shortage-dependent partial backlogging with capacity constraint", *International Journal of Systems Science*, vol. 40, (2009), pp. 33-44.
- [13] C. Lin, B. Tan and W. -C. Lee, "An EOQ model for deteriorating items with time-varying demand and shortages", *International Journal of Systems Science*, vol. 31, (2000), pp. 391-400.
- [14] B. C. Giri, S. Pal, A. Goswami and K. S. Chaudhuri, "An inventory model for deteriorating items with stock-dependent demand rate", *European Journal of Operational Research*, vol. 95, (1996), pp. 604-610.
- [15] B. C. Giri and K. S. Chaudhuri, "Deterministic models of perishable inventory with stock dependent demand rate and nonlinear holding cost", *European Journal of Operational Research*, vol. 105, (1998), pp. 467-474.
- [16] S. K. Goyal and B. C. Giri, "The production-inventory problem of a product with time varying demand, production and deterioration rates", *European Journal of Operational Research*, vol. 147, (2003), pp. 549-557.
- [17] K. -S. Wu, "EOQ inventory model for items with Weibull distribution deterioration, time-varying demand and partial backlogging", *International Journal of Systems Science*, vol. 33, (2002), pp. 323-329.
- [18] N. H. Shah and K. T. Shukla, "Deteriorating inventory model for waiting time partial backlogging", *Applied Mathematical Sciences*, vol. 3, (2009), pp. 421-428.
- [19] K. Skouri, I. Konstantaras, S. Papachristos and I. Ganas, "Inventory models with ramp type demand rate, partial backlogging and Weibull deterioration rate", *European Journal of Operational Research*, vol. 192, (2009), pp. 79-92.
- [20] H. J. Chang and C. -Y. Dye, "An inventory model for deteriorating items with partial backlogging and permissible delay in payments", *International Journal of Systems Science*, vol. 32, (2001), pp. 345-352.
- [21] H. -J. Chang, C. -H. Hung and C. -Y. Dye, "A finite time horizon inventory model with deterioration and time-value of money under the conditions of permissible delay in payments", *International Journal of Systems Science*, vol. 33, (2002), pp. 141-151.
- [22] K. -L. Hou and L. -C. Lin, "An EOQ model for deteriorating items with price- and stock-dependent selling rates under inflation and time value of money", *International Journal of Systems Science*, vol. 37, (2006), pp. 1131-1139.
- [23] M. Basu and S. Sinha, "An inflationary inventory model with time dependent demand with Weibull distribution deterioration and partial backlogging under permissible delay in payments", *Control and Cybernetics*, vol. 36, (2007), pp. 203-217.
- [24] I. Moon, B. C. Giri and B. Ko, "Economic order quantity models for ameliorating/deteriorating items under inflation and time discounting", *European Journal of Operational Research*, vol. 162, (2005), pp. 773-785.

- [25] N. H. Shah, "Inventory model for deteriorating items and time value of money for a finite time horizon under the permissible delay in payments", *International Journal of Systems Science*, vol. 37, (2006), pp. 9-15.
- [26] S. Nahmias, "Optimal ordering policies for perishable inventory", *Operations Research*, vol. 23, (1975), pp. 735-749.
- [27] S. Nahmias, "On ordering perishable inventory under Erlang demand", *Naval Research Logistics Quarterly*, vol. 22, (1978), pp. 415-425.
- [28] W. Pierskalla, "Optimal issuing policies in inventory management", *Management Science*, vol. 13, (1967), pp. 1313-1317.
- [29] W. Pierskalla and C. D. Roach, "Optimal issuing policies for perishable inventory", *Management Science*, vol. 18, (1972), pp. 603-614.
- [30] P. M. Ghare and G. F. Schrader, "A model for exponential decaying inventory", *Journal of Engineering*, vol. 14, (1963), pp. 238-243.
- [31] R. H. Wilson, "A scientific routine for stock control", *Harvard Business Review*, vol. 13, (1934), pp. 116-128.
- [32] U. Dave and L. K. Patel, "(T,Si) policy inventory model for deteriorating items with time proportional demand", *Journal of Operational Research Society*, vol. 32, (1981), pp. 137-142.
- [33] R. S. Sachan, "On (T,Si) inventory policy model for deteriorating items with time proportional demand", *Journal of Operational Research Society*, vol. 35, (1984), pp. 1013-1019.
- [34] A. Goswami and K. S. Chaudhuri, "An EOQ model for deteriorating items with shortages and a linear trend in demand", *Journal of Operational Research Society*, vol. 42, (1991), pp. 1105-1110.
- [35] M. Hariga, "Optimal EOQ models for deteriorating items with time-varying demand", *Journal of Operational Research Society*, vol. 47, (1996), pp. 205-213.
- [36] K. -S. Wu, "An EOQ inventory model for items with Weibull distribution deterioration, ramp type demand rate and partial backlogging", *Production Planning and Control*, vol. 12, (2001), pp. 787-793.
- [37] H. J. Chang and C. -Y. Dye, "An EOQ model for deteriorating items with time varying demand and partial backlogging", *Journal of Operational Research Society*, vol. 50, (1999), pp. 1176-1182.
- [38] C. -Y. Dye, "Joint pricing and ordering policy for a deteriorating inventory with partial backlogging", *Omega-The International Journal of Management Science*, vol. 35, (2007), pp. 184-189.
- [39] S. K. Goyal and B. C. Giri, "Recent trends in modeling of deteriorating inventory", *European Journal of Operational Research*, vol. 134, (2001), pp. 1-16.
- [40] K. Sunil and M. Swaminathan, "Diffusion of innovations under supply constraints", *Operations Research*, vol. 51, (2003), pp. 866-879.

## Authors



**Dr. Xiaoming Yan** received the B.S. degree in Applied Mathematics from Zhejiang Normal University, Jinhua, Zhejiang, P.R. China in 2004, and the Ph.D. degree in Operations Research from the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2009. He is now an Assistant Professor in the School of Computer at the Dongguan University of Technology. His email address is yanxiaoming325@126.com.